Graph Mining Class Notes

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1 Notes

Definition 1. (Graph Density) *Let H be a graph with V vertices and E edges, H = (V,E). The density of graph H is ρ =E/V.*

Definition 2. (Balanced Graph) *We call H is balanced if $\forall$ subgraph $H\prime \subseteq H$, $\rho(H\prime) \leq \rho(H)$.*

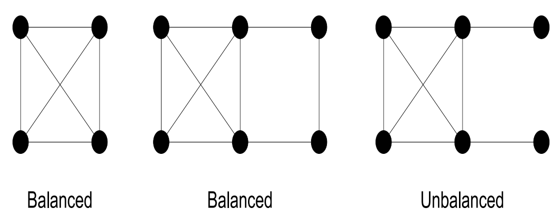

Fig. 1. Balanced Graph Example

Figure 1, shows an example on balanced graph which first two are balanced but the third one is not.

Definition 3. (Strictly Balanced Graph) *We call H is strictly balanced if $\forall$ subgraph $H\prime \subseteq H$, $\rho(H\prime) < \rho(H)$.*

Definition 4. *Let H is a balanced graph, H = (V,E). Let $A(G)$ be the even that H is a subgraph of $G_{n,p}$. $[G_{n,p}] = A(G)$*

For $G_{n,p}$, which n is number of vertices and p is the probability, has $2^{\binom{n}{2}}$ subgraphs. Let X is the number of balanced subgraphs.

$$\begin{aligned} EX &= \tbinom{n}{v} \times EX_i \\ &= \tbinom{n}{v} \times P(H \subseteq X_i) \end{aligned}$$

Lemma 1. $P^e \leq P(H \subseteq X_i) \leq v! \times P^e$

$$\Longrightarrow \binom{n}{v} \times P^e \leq EX \leq \binom{n}{v} \times v! \times P^e$$
$$\Longrightarrow P = \ominus(n^{-\frac{v}{e}})$$

For

$$\binom{n}{v} \times P^e \approx \ominus(n^v \times P^e)$$
$$EX \approx \ominus(1)$$
$$\binom{n}{v} \times v! \times P^e \approx (n^v \times P^e) = 1$$

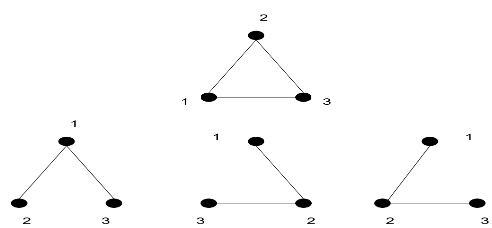

Fig. 2. Lemma Example

In Lemma1, $v!$ means the order of vertices could be changed while is still the same graph. This definition is similar as graph isomorphism. To prove lemma1, the simplest situation is that, graph H is a clique. In figure 2, the above triangle is p^3 and all other triangle are $3 \times p^2$.

Proof. There are two situations:

$$r(n) \ll n^{-\frac{v}{e}}, p(x > 0) \longrightarrow 0 \quad (1)$$
$$r(n) \gg n^{-\frac{v}{e}}, p(x < 1) \longrightarrow 0 \quad (2)$$

For situation (1):

$$p(x > 0) \leq EX = n^v \times r(n)^e \longrightarrow 0$$

For situation (2):

$$\begin{aligned} p(x < 1) &= p(x = 0) && \{x : integer\} \\ &\leq \frac{VarX}{(EX)^2} && \{ChebyshevInequlity\} \\ &\longrightarrow 0 \end{aligned}$$

1. "$VarX$"

$$
\begin{aligned}
VarX = Var(\sum_{i=1}^{\binom{n}{v}} X_i) &= E((\sum_{i=1}^{\binom{n}{v}} X_i) - E(\sum_{i=1}^{\binom{n}{v}} X_i))^2 \\
&= \sum_{i=1}^{\binom{n}{v}} VarX_i + \sum_{i=1}^{\binom{n}{v}} \sum_{i \neq j} Cov(X_i, X_j)
\end{aligned}
$$

$$
\begin{aligned}
VarX_i &= EX_i^2 - (EX_i)^2 \\
&= P(X_i = 1) - P^2(X_i = 1) \\
&\leq P(X_i = 1) \\
&= EX_i
\end{aligned}
$$

$$
\sum_{i=1}^{\binom{n}{v}} VarX_i \leq \binom{n}{v} \times EX_i = EX
$$

$$
\Longrightarrow VarX \leq EX + \sum_{i \neq j} Cov(X_i, X_j)
$$

1-1. “EX”

$$
for \;\; EX = \binom{n}{v} \times P^e \approx n^v \times P^e
$$

$$
when \;\; P \gg P(n) = n^{-\frac{v}{e}}
$$

$$
\Longrightarrow EX \longrightarrow +\infty
$$

1-2. “$\sum_{i \neq j} Cov(X_i, X_j)$”

$$
(1) \;\; |V_{X_i} \cap V_{X_j}| \leq 1 \;\; \{independant\}
$$

$$
Cov(X_i, X_j) = 0
$$

$$(2)\ \ |V_{X_i} \cap V_{X_j}| \geq 2\ \ \{dependant\}$$

$$\begin{aligned}
\sum_{i\neq j} Cov(X_i, X_j) &= \sum_{i\neq j}(E(X_i, X_j) - EX_i \times EX_j) \\
&\leq \sum_{i\neq j} E(X_i, X_j) - \sum_{i\neq j} P(X_i = 1 \cap X_j = 1) \\
&= \sum_{i\neq j} P(X_j = 1 | X_i = 1) \times P(X_i = 1) \\
&= \sum_{i=1}^{\binom{n}{v}} P(X_i = 1) \times \sum_{i\neq j} P(X_j = 1 | X_i = 1)
\end{aligned}$$

Let $\triangle^* : \sum_{i\neq j} P(X_j = 1 | X_i = 1)$,

$$\begin{aligned}
\triangle^* &= \sum_{i\neq j} P(X_j = 1 | X_i = 1) \\
&= \sum_{k=|V_{X_i} \cap V_{X_j}|=2}^{v-1} \binom{v}{k} \times \binom{n-v}{v-k} \times P(X_j = 1 | X_i = 1) \\
&\leq \sum_{k=|V_{X_i} \cap V_{X_j}|=2}^{v-1} \binom{v}{k} \times \binom{n-v}{v-k} \times P^{e - \frac{v}{k \times e}} \times (v-k)! \\
&\leq \sum_{k=|V_{X_i} \cap V_{X_j}|=2}^{v-1} n^{v-k} \times P^{e - \frac{v}{k \times e}} \\
&= \sum_{k=|V_{X_i} \cap V_{X_j}|=2}^{v-1} (n^v \times P^e)^{1 - \frac{v}{k}} \\
&\approx \bigcirc(n^v \times P^e)
\end{aligned}$$

$$\begin{aligned}
\longrightarrow \sum_{i\neq j} Cov(X_i, X_j) &= \sum_{i=1}^{\binom{n}{v}} P(X_i = 1) \times \triangle^* \\
&= EX \times \triangle^*
\end{aligned}$$

$$\begin{aligned}
So:\ \ VarX &\leq EX + \sum_{i\neq j} Cov(X_i, X_j) \\
&= EX + EX \times \triangle^*
\end{aligned}$$

2. "$\frac{1}{EX}$"

$$for\ \ EX = \binom{n}{v} \times P^e \approx n^v \times P^e$$

$$when\ \ P \gg P(n) = n^{-\frac{v}{e}}$$

$$\Longrightarrow EX \longrightarrow +\infty$$

$$\Longrightarrow \frac{1}{EX} \longrightarrow 0$$

$$InConclusion:$$
$$\frac{EX^2}{VarX} \leq (\frac{EX}{\triangle^*}) \longrightarrow 0$$