Logic and Proof

Argument

An argument is a sequence of statements.

All statements but the first one are called assumptions or hypothesis.

The final statement is called the conclusion.

An argument is valid if:

> whenever all the assumptions are true, then the conclusion is true.

If today is Wednesday, then yesterday is Tuesday.

Today is Wednesday.

$\therefore$ Yesterday is Tuesday.

Modus Ponens

If p then q.
p
∴ q

p	q	p→q	p	q

Modus ponens is Latin meaning "method of affirming".

Modus Tollens

If p then q.
~q
∴ ~p

p	q	p→q	~q	~p

Modus tollens is Latin meaning “method of denying”.

Equivalence

A student is trying to prove that propositions *P*, *Q*, and *R* are all true. She proceeds as follows.

First, she proves three facts:

- *P* implies *Q*
- Q implies *R*
- *R* implies *P*.

Then she concludes,

``Thus *P*, *Q*, and *R* are all true.''

Proposed argument:

$$\frac{(P \to Q),\ (Q \to R),\ (R \to P)}{P \wedge Q \wedge R}$$

Is it valid?

Valid Argument?

Conclusion true whenever all assumptions are true.

assumptions

conclusion

To prove an argument is not valid, we just need to find a counterexample.

Valid Arguments?

If p then q.
q
$\therefore$ p

If you are a fish, then you drink water.
You drink water.
You are a fish.

If p then q.
~p
$\therefore$ ~q

If you are a fish, then you drink water.
You are not a fish.
You do not drink water.

Exercises

$$\begin{array}{l} \quad p \\ \therefore\ p \vee q \end{array}$$

$$\begin{array}{l} \quad p \wedge q \\ \therefore\ p \end{array}$$

$$\begin{array}{l} \quad p \vee q \\ \quad \neg q \\ \therefore\ p \end{array}$$

$$\begin{array}{l} \quad p \\ \therefore\ p \wedge q \end{array}$$

$$\begin{array}{l} \quad p \vee q \\ \therefore\ p \end{array}$$

$$\begin{array}{l} \quad p \longrightarrow q \\ \quad q \longrightarrow r \\ \therefore\ p \longrightarrow r \end{array}$$

More Exercises

$$\neg p \rightarrow q$$
$$\neg q$$
$$\therefore\ p$$

$$\neg p \rightarrow \neg q$$
$$\therefore\ p \rightarrow q$$

$$\neg p \rightarrow \neg q$$
$$\therefore\ q \rightarrow p$$

$$1 = -1$$
$$\therefore\ \text{Today is Tuesday.}$$

Valid argument $\not\rightarrow$ True conclusion

True conclusion $\not\rightarrow$ Valid argument

Contradiction

$$\neg p \longrightarrow c$$
$$\therefore\ p$$

If you can show that the assumption that the statement p is false leads logically to a contradiction, then you can conclude that p is true.

You are working as a clerk.

If you have won Mark 6, then you would not work as a clerk.

$\therefore$ You have not won Mark 6.

Arguments with Quantified Statements

Universal instantiation:

$$\forall x, P(x)$$
$$P(a)$$

Universal modus ponens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$P(a)$$
$$\therefore\ Q(a)$$

Universal modus tollens:

$$\forall x, P(x) \rightarrow Q(x)$$
$$\neg Q(a)$$
$$\therefore\ \neg P(a)$$

Universal Generalization

valid rule

$$\frac{A \rightarrow R(c)}{A \rightarrow \forall x.R(x)}$$

providing c is independent of A

e.g. given any number x, 2x is an even number

=> for all x, 2x is an even number.

Not Valid

$$\forall z\,[Q(z) \vee P(z)] \rightarrow [\forall x.Q(x) \vee \forall y.P(y)]$$

Proof: Give ***countermodel***, where

$\forall z\,[Q(z) \vee P(z)]$ is true,

but $\forall x.Q(x) \vee \forall y.P(y)$ is false.

Find a domain, and a predicate.

In this example, let domain be integers,

$Q(z)$ be true if z is an even number, i.e. Q(z)=even(z)

$P(z)$ be true if z is an odd number, i.e. P(z)=odd(z)

Validity

$$\forall z\,[Q(z) \wedge P(z)] \rightarrow [\forall x.Q(x) \wedge \forall y.P(y)]$$

Proof strategy: We assume $\forall z\,[Q(z) \wedge P(z)]$

and prove $\forall x.Q(x) \wedge \forall y.P(y)$

Proof and Logic

We prove mathematical statement by using logic.

$$\frac{P \rightarrow Q,\ \ Q \rightarrow R,\ \ R \rightarrow P}{P \wedge Q \wedge R}$$

not valid

To prove something is true, we need to assume some **axioms**!

This is invented by Euclid in 300 BC,
who begins with 5 assumptions about geometry,
and derive many theorems as logical consequences.

http://en.wikipedia.org/wiki/Euclidean_geometry

Proofs

Proving an Implication

Goal: If P, then Q. (P implies Q)

Method 1: Write assume P, then show that Q logically follows.

Claim: If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$

Proving an Implication

Goal: If P, then Q. (P implies Q)

Method 1: Write assume P, then show that Q logically follows.

Claim: If r is irrational, then $\sqrt{r}$ is irrational.

How to begin with?

What if I prove "If $\sqrt{r}$ is rational, then r is rational", is it equivalent?

Yes, this is equivalent;

proving "if P, then Q" is equivalent to proving "if not Q, then not P".

Proving an Implication

Goal: If P, then Q. (P implies Q)

Method 2: Prove the ***contrapositive***, i.e. prove "not Q implies not P".

Claim: If r is irrational, then $\sqrt{r}$ is irrational.

Proving an "if and only if"

Goal: Prove that two statements P and Q are "**logically equivalent**", that is, one holds if and only if the other holds.

Example:

An integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3.

Method 1: Prove P implies Q **and** Q implies P.

Method 1': Prove P implies Q **and** not P implies not Q.

Method 2: Construct a chain of if and only if statement.

Proof the Contrapositive

Statement: If m^2 is even, then m is even

Try to prove directly.

Proof the Contrapositive

Statement: If m^2 is even, then m is even

Contrapositive: If m is odd, then m^2 is odd.

Proof (the contrapositive):

Proof by Contradiction

$$\frac{\bar{P} \rightarrow \mathrm{F}}{P}$$

To prove P, you prove that not P would lead to ridiculous result,
and so P must be true.

You are working as a clerk.

If you have won Mark 6, then you would not work as a clerk.

$\therefore$ You have not won Mark 6.

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

- Suppose $\sqrt{2}$ was rational.
- Choose *m*, *n* integers without common prime factors (always possible)

  such that $$\sqrt{2} = \frac{m}{n}$$

- Show that *m* and *n* are both even, thus having a common factor 2,

  a **contradiction**!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

Want to prove both m and n are even.

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

Want to prove both m and n are even.

$$\sqrt{2} = \frac{m}{n}$$

$$\sqrt{2}n = m$$

$$2n^2 = m^2$$

so m is even.

so can assume $m = 2l$

$$m^2 = 4l^2$$

$$2n^2 = 4l^2$$

$$n^2 = 2l^2$$

so n is even.

Proof by Cases

$$p \vee q$$
$$p \rightarrow r$$
$$q \rightarrow r$$
$$\therefore\ r$$

e.g. want to prove a nonzero number always has a positive square.

x is positive or x is negative

if x is positive, then $x^2 > 0$.

if x is negative, then $x^2 > 0$.

$\therefore$ $x^2 > 0$.

Rational vs Irrational

Question: If a and b are irrational, can a^b be rational??

We know that $\sqrt{2}$ is irrational, what about $\sqrt{2}^{\sqrt{2}}$?

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational

So in either case there are a,b irrational and a^b be rational.

We don't need to know which case is true!

Extra

Power and Limits of Logic

Good news: Gödel's Completeness Theorem

Only need to know a few axioms & rules, to prove *all* validities.

That is, starting from a few propositional & simple predicate validities, every valid assertion can be proved using just universal generalization and *modus ponens* repeatedly!

modus ponens

$$\frac{P \to Q, P}{Q}$$

Power and Limits of Logic

Thm 2, bad news:

Given a set of axioms,

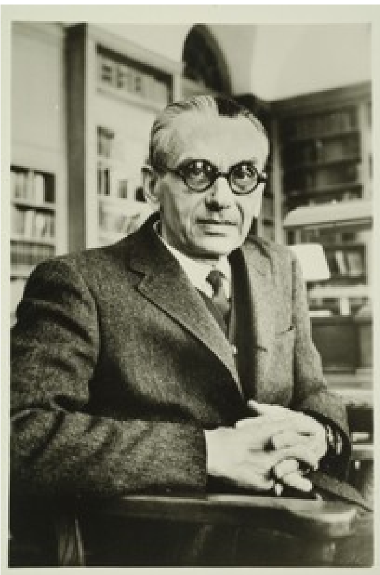
there is *no procedure* that decides

whether quantified assertions are valid.

(unlike propositional formulas).

Power and Limits of Logic

Gödel's *In*completeness Theorem for Arithmetic

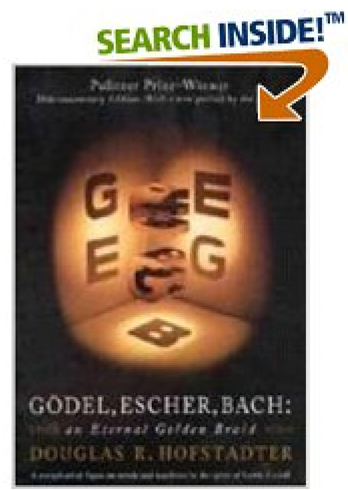


Thm 3, worse news:

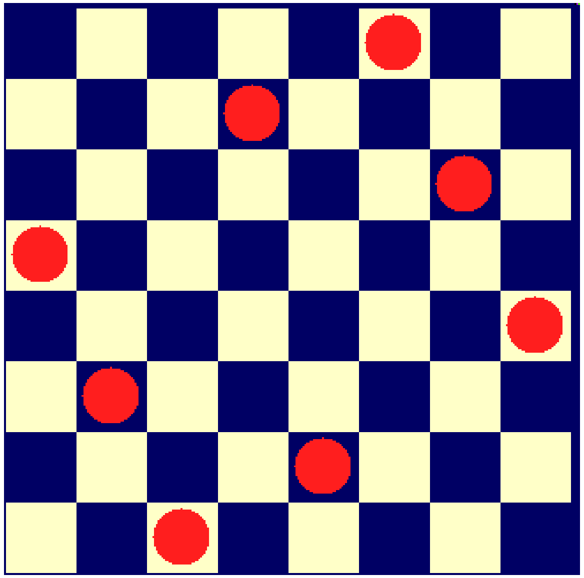
For any "reasonable" theory that proves basic arithmetic truth, an arithmetic statement that is true, but not provable in the theory, can be constructed.

No hope to find a complete and consistent set of axioms!

An excellent project topic:



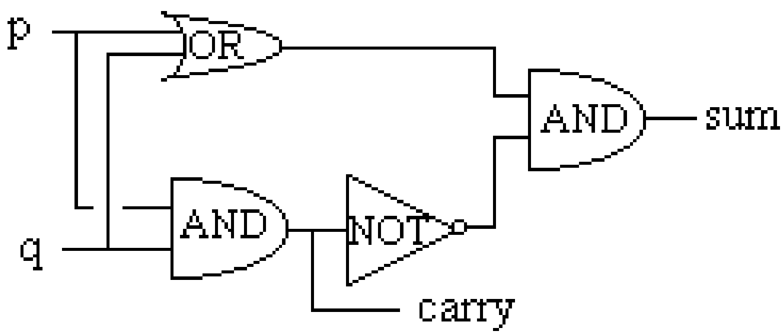
Application: Logic Programming



	1		6		7			4
	4	2						
8	7		3			6		
	8			7			2	
			8	9	3			
	3			6			1	
		8			6		4	5
						1	7	
4			9		8		6	

Other Applications

Digital logic:



p	q	sum	carry
1	1	0	1
1	0	1	0
0	1	1	0
0	0	0	0

Database system:

Making queries

Data mining