Some Key Ideas from Chapter 2

Section 2.1 Limits and Continuity

The statement

$$lim_{x\to c}f(x) = L$$

means that the values $f(x)$ are arbitrarily close to L when the values x are sufficiently close to c. Please note that $lim_{x\to c}f(x) = L$ does not depend on $f(c)$ at all. $f(c)$ may or may not exist, and whether or not it does exists, does not affect $lim_{x\to c}f(x) = L$.

We can use tables to find $lim_{x\to c}f(x) = L$; see Example 1, pages 78 and 79. We can also use graphs; see the last part of Example 2, page 80.

The idea of the limit of a function as the domain values approach a fixed value can be further broken down to left-hand and right-hand limits. (Remember that this fixed value may be but need not be in the domain of the function, i.e., the function need not be defined at the fixed value.) The expression "left-hand limit" means that the domain values which are approaching the fixed value, i.e., our c, are less than c, i.e., they can be thought of as approaching from the "left side" of c when one thinks of the real number line running from left - the "negative end" - to the right - the "positive end". Likewise, the expression "right-hand limit" means that the domain values which are approaching c are greater than c. We thus can have the following two expressions:

$$lim_{x\to c^-}f(x) = L$$

and

$$lim_{x\to c^+}f(x) = L.$$

Given a function f and a fixed value c, we have three possible limits. They are $lim_{x\to c}f(x)$, $lim_{x\to c^-}f(x)$, and $lim_{x\to c^+}f(x)$. It may be the case that none of these limits exists, or all may exist. And, of course, there are other possibilities. If all three exist, then they must all be equal. In fact, for $lim_{x\to c}f(x)$ to exist, then $lim_{x\to c^-}f(x)$ and $lim_{x\to c^+}f(x)$ must both exist and be equal, and then $lim_{x\to c}f(x)$ equals their common value. If either $lim_{x\to c^-}f(x)$ or $lim_{x\to c^+}f(x)$ does not exist or if they both exist but are not equal, then $lim_{x\to c}f(x)$ can not exist because $lim_{x\to c}f(x)$ only exists when $lim_{x\to c^-}f(x) = lim_{x\to c^+}f(x)$, and then

$$lim_{x\to c}f(x) = lim_{x\to c^-}f(x) = lim_{x\to c^+}f(x).$$

If we had to find all limits by building tables or drawing graphs or doing limit calculations (see Example 1, pages 95 and 96 - in Section 2.2), then working with limits could be difficult. However, finding limits is often relatively easy because our functions are often continuous functions.

A function f is continuous at c if

1. $f(c)$ is defined, i.e., c is in the domain of f,
2. $lim_{x \to c} f(x)$ exists, and
3. $lim_{x \to c} f(x) = f(c)$, i.e., $L = f(c)$.

Part 3 is what really helps us. When we are working with a function f which is continuous at c, then the L in $lim_{x \to c} f(x) = L$ is just $f(c)$. Thus, we simply evaluate f at c and we have our limit!

All polynomials, all rational functions, all exponential functions, and all logarithmic functions are continuous whenever they are defined, i.e., are continuous for every value in their domains.

The ∞ symbol is used with limits. Look at exercise 50 on page 91. For this exercise we can write the following:

$$lim_{x \to -3^-} f(x) = \infty$$

$$lim_{x \to -3^+} f(x) = -\infty$$

$$lim_{x \to -\infty} f(x) = 1$$

$$lim_{x \to \infty} f(x) = 1$$

For this exercise, we would clearly say that $lim_{x \to -3} f(x)$ does not exist, and we are correct. However, it is also that case that neither $lim_{x \to -3^-} f(x)$ nor $lim_{x \to -3^+} f(x)$ exists. Whenever we write a limit or a left-hand or right-hand limit equals $-\infty$ or ∞, then we are saying two things. We are saying the (one-sided) limit does not exist, and we are saying why it does not exist - because either the f-values are decreasing or increasing without bound. To see the difference in saying $lim_{x \to c} f(x)$ does not exist and in saying $lim_{x \to c} f(x) = \infty$ also consider exercise 51 on page 91. In exercise 50, we can only say that $lim_{x \to -3} f(x)$ does not exist. However in exercise 51, we can say $lim_{x \to -2} f(x) = \infty$. In neither case does the limit exist, but for exercise 51, we are also saying why it does not exist.

When we write $lim_{x \to -\infty} f(x) = 1$ or $lim_{x \to \infty} f(x) = 1$, we are saying that the limit exists. We can say this because the limit is a real number, i.e., it is 1. However, "∞" is not a real number. Thus, writing $lim_{x \to -2} f(x) = \infty$ for exercise 51 says the limit does not exist, and further, the limit doesn't exist because the values of $f(x)$ increase without bound as x approaches -2.