Signal Processing for Visualization

Materials from Prof. Klaus Mueller
Signal Processing

• Analysis, interpretation and manipulation of signals

• Scientific Visualization:
  – Scientific datasets → digital signals
  – We mainly focus on signal sampling and reconstruction
    • Correctly represent the data
    • Visualize the data in a correct way

• **Sampling**: reduction of continuous signals to discrete signals

• **Reconstruction**: recovery of the signals from discrete samples
Fourier Transform

• A typical method used in visualization to explain data behavior
• A mathematical method named in honor of French mathematician Joseph Fourier
• Decomposes a function into a continuous spectrum of its frequency components
• The inverse transform synthesizes a function from its spectrum of frequency components.
General Definition

- $X(t)$: Frequency domain function of continuous function $x(t)$

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} \, dt,
\]

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} \, df,
\]

- Please refer to Foley et al. book page 624 or other signal processing references
Fourier Domain

- The frequency domain is complement to spatial or temporal domain
- Filter design more easy in Fourier domain than in the spatial domain
- Easy to understand artifacts or aliasing in visualization
Usage of Fourier Domain

- Analyze the frequency of spatial or temporal data sets
- Determine how many samples should be used to discretize continuous signals
- Design proper filters to reconstruct data for proper rendering
  - Avoid aliasing
- However, besides the Fourier domain analysis, actual data operations usually performed on spatial or temporal domain
Fourier Expansion

- Map continuous signal to a Fourier series of samples
- A sine function

\[ f(t) = A \cdot \sin(2\pi \omega t + \phi) \]

- A: amplitude
- \( \omega \): frequency
- \( \phi \): phase
- t: time
Fourier Expansion

- Any periodic continuous function can be represented by a linear combination of sine functions

\[ f(t) = A_0 + \sum_{k=1}^{N} A_k \sin(2\pi k \omega_0 t + \varphi_k) \]

- This is Fourier series expansion
  - \( \omega_0 \) is the fundamental frequency of the continuous signal
  - Only frequencies \( k \omega_0 \) are used (\( k \) is int)
  - They are harmonics

- Periodic but discontinuous signals can only be approximated by a Fourier Series
Fourier Series

- Periodic continuous signals are transformed into Fourier series
  - Lower harmonics yield the rough shape of the signal
  - Higher harmonics generate the sharpness of the edges (high contrasts)
Fourier Transform
Square Wave

spatial / temporal domain:

Fourier / frequency domain:

\[ F_k = \frac{A d}{T} \frac{\sin\left(\frac{k \omega_0 d}{2}\right)}{k \omega_0 d} = \frac{A d}{T} \text{sinc}\left(\frac{k \omega_0 d}{2}\right) \]

sinc-envelope
Increase Square Wave Gaps

- Square wave moves apart
- More harmonics will be used
Non-periodic Signal

- Square wave signal becomes non-periodic.
- Transforms to continuous sinc function
- \( \text{sinc}: \) continuous Fourier spectrum
Fourier Transform Pair

- $F(\omega)$ is the amplitude of the signal at frequency $\omega$
- $f(t)$ is the amplitude of the signal at time $t$

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \quad \text{Fourier Transform (FT)}
\]

\[
f(t) = \int_{-\infty}^{\infty} F(\omega) e^{2\pi i \omega t} d\omega \quad \text{Inverse Fourier Transform (IFT)}
\]
Fourier Transform Pair

- rect(x) $\Leftrightarrow$ sinc(\(\omega\))

\[
rect_d(t) = \begin{cases} 
\frac{1}{d} & |t| \leq \frac{d}{2} \\
0 & \text{elsewhere}
\end{cases}
\]

$\Leftrightarrow \frac{\sin\left(\frac{\pi \omega}{d}\right)}{\frac{\pi \omega}{d}} = \text{sinc}\left(\frac{\pi \omega}{d}\right)$

- Finite signal transforms to infinite response in Fourier domain
Fourier Transform Pair

- Dirac delta function
- Impulse function
- A infinite pulse at $x=0$

\[ \cos(2\pi at) \Leftrightarrow \frac{\delta(\omega + a) + \delta(\omega - a)}{2} \]

- Transforms to cosine function
Sampling and reconstruction

Input signal (cont.) → Sampling procedure → Discrete samples

Sample trail

Discrete samples → Reconstruction procedure → Output signal (cont.)

Reconstruction filter
Sampling

- A continuous signal is measured at fixed instances by an interval T
  - Sampling period T
  - $1/T$ Sampling frequency

- The data samples obtained form a discrete signal
Sampling with Impulses

- This sampling procedure can be done with a series of Dirac delta impulses.
Sampling and Fourier Domain

- A series of impulses $\iff$ a series of impulse train of period $\omega_s = 1/T$ in Fourier domain (frequency domain)

- Closer sampling points $\Rightarrow$ wider apart of spikes in frequency domain
Sampling

• Sampling replicates origin signal spectrum $X_c$ at integer multiples of $\omega_s$
Aliasing Occurrence

- If replication overlaps in Fourier domain

main spectrum = $X_c(\omega)$

$X_s(\omega)$

side spectra = copies (aliases) of $X_c(\omega)$

aliased $X_s(\omega)$
Signal Reconstruction

- Reconstruct original continuous signal from sampling points
- This is actually recover the main spectrum and remove side spectra
- If main spectrum and side spectra overlap
  - Cannot recover correct signal
  - Aliasing happens
Example

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 0.5 (samples per time unit) → original signal can not be recovered

Sample points $x[n]$

Original signal $x_c$

Aliased signal $x_{c_{\text{aliased}}}$ reconstructed from the sample points $x[n]$
Example

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 0.7 (samples per time unit)
- Looking at the sample points $x[n]$, they appear to originate from a sine wave $x_{c_{\text{aliased}}}$ of much lower frequency → again, the original sine wave is lost and cannot be recovered

sample points $x[n]$

original signal $x_c$

aliased signal $x_{c_{\text{aliased}}}$

reconstructed from the sample points $x[n]$
Example

- Frequency of original signal: 0.5 (oscillations per time unit)
- Sampling frequency: 1.0 (sample per time unit) → original signal can be recovered
- We learn that we need to sample each oscillation period twice for good reconstruction

Sample points $x[n]$

Original signal $x_c$

Non-aliased signal $x_{c\_non\_aliased}$

Reconstructed from the sample points $x[n]$
Nyquist Frequency

• Sampling theorem: A band-limited signal can be **EXACTLY** reconstructed from its samples if sampling frequency is at least twice the maximum signal frequency

• This minimum required sampling frequency is called Nyquist frequency

\[ \omega_N = 2\omega_{\text{max}} \]

• Intuition: two samples per period are needed to resolve oscillation
Aliasing Prevention 1

- Increase sampling frequency
- Short sampling interval
- In practical, try to use a sampling rate slightly higher than Nyquist frequency
Aliasing Prevention 2

• When high sampling rate is not feasible
  – For example, sinc has an infinite spectrum
  – Sampling rate needed is too high to be applicable in reality

• Solution
  – Use limited signal maximum frequency
  – Low pass filter
Low Pass Filter

- Filter (or smooth high frequency signals)
- Image smoothing as we discussed
  - Remove sharp edges
  - Blurred result
- This anti-aliasing method loses some original features
Image: 2D Fourier Transform

- Method:

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i (ux + vy)} \, dx \, dy \]  
2D Fourier Transform (FT)

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi i (ux + vy)} \, du \, dv \]  
Inverse 2D Fourier Transform (IFT)

- 2D frequency spectrum
Image Sampling

• Take a photo $\rightarrow$ sample continuous real world scene into a digital image
Image Anti-aliasing

- If \( u_{\text{max}} \) and \( v_{\text{max}} \) are maximal frequencies in \( x \) and \( y \) directions

- To avoid aliasing
  - \( \frac{1}{T_x} > 2 \ u_{\text{max}} \)
  - \( \frac{1}{T_y} > 2 \ v_{\text{max}} \)

- In reconstruction, isolate main spectrum from side spectra
Image Anti-aliasing

- More options
  - Choose high enough sampling rate
  - Low pass the image
- Low pass results
  - Eliminate sharpness
  - Blurred image

original  blurred after lowpassing
• Same strategy
• We will discuss more