Nested Quantifiers

Section 1.5
Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

**Example:** “Every real number has an inverse” is

$$\forall x \exists y (x + y = 0)$$

where the domains of $x$ and $y$ are the real numbers.

We can also think of nested propositional functions:

$$\forall x \exists y (x + y = 0)$$ can be viewed as $\forall x \ Q(x)$ where $Q(x)$ is

$$\exists y \ P(x, y)$$ where $P(x, y)$ is $(x + y = 0)$
Thinking of Nested Quantification

- **Nested Loops**
  - To see if $\forall x \forall y P(x, y)$ is true, loop through the values of $x$:
    - At each step, loop through the values for $y$.
    - If for some pair of $x$ and $y$, $P(x, y)$ is false, then $\forall x \forall y P(x, y)$ is false and both the outer and inner loop terminate.
  
  $\forall x \forall y P(x, y)$ is true if the outer loop ends after stepping through each $x$.
  
  - To see if $\forall x \exists y P(x, y)$ is true, loop through the values of $x$:
    - At each step, loop through the values for $y$.
    - The inner loop ends when a pair $x$ and $y$ is found such that $P(x, y)$ is true.
    - If no $y$ is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x, y)$ has been shown to be false.
  
  $\forall x \exists y P(x, y)$ is true if the outer loop ends after stepping through each $x$.

- If the domains of the variables are infinite, then this process can not actually be carried out.
Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement “$x + y = y + x.$” Assume that $U$ is the real numbers. Then $\forall x \ \forall y P(x,y)$ and $\forall y \ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement “$x + y = 0.$” Assume that $U$ is the real numbers. Then $\forall x \ \exists y P(x,y)$ is true, but $\exists y \ \forall x P(x,y)$ is false.
Questions on Order of Quantifiers

Example 1: Let $U$ be the real numbers, Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$
   Answer: False

2. $\forall x \exists y P(x,y)$
   Answer: True

3. $\exists x \forall y P(x,y)$
   Answer: True

4. $\exists x \exists y P(x,y)$
   Answer: True
Example 2: Let $U$ be the real numbers without 0, Define $P(x,y) : x/y = 1$
What is the truth value of the following:

1. $\forall x \forall y P(x,y)$  
   Answer: False

2. $\forall x \exists y P(x,y)$  
   Answer: True

3. $\exists x \forall y P(x,y)$  
   Answer: False

4. $\exists x \exists y P(x,y)$  
   Answer: True
### Quantifications of Two Variables

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \forall y P(x, y)$</td>
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Translating Nested Quantifiers into English

**Example 1:** Translate the statement

\[ \forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y))) \]

where \(C(x)\) is “\(x\) has a computer,” and \(F(x,y)\) is “\(x\) and \(y\) are friends,” and the domain for both \(x\) and \(y\) consists of all students in your school.

**Solution:** Every student in your school has a computer or has a friend who has a computer.

**Example 1:** Translate the statement

\[ \exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y, z)) \]

**Solution:** There is a student none of whose friends are also friends with each other.
Example: Translate “The sum of two positive integers is always positive” into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   “For every two integers, if these integers are both positive, then the sum of these integers is positive.”

2. Introduce the variables $x$ and $y$, and specify the domain, to obtain:
   “For all positive integers $x$ and $y$, $x + y$ is positive.”

3. The result is:
   $$\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$
   where the domain of both variables consists of all integers
Example: Use quantifiers to express the statement “There is a woman who has taken a flight on every airline in the world.”

Solution:

1. Let $P(w,f)$ be “$w$ has taken $f$” and $Q(f,a)$ be “$f$ is a flight on $a$.”
2. The domain of $w$ is all women, the domain of $f$ is all flights, and the domain of $a$ is all airlines.
3. Then the statement can be expressed as:
   \[ \exists w \ \forall a \ \exists f \ (P(w,f) \land Q(f,a)) \]
Choose the obvious predicates and express in predicate logic.

Example 1: “Brothers are siblings.”
Solution: $\forall x \forall y (B(x,y) \rightarrow S(x,y))$

Example 2: “Siblinghood is symmetric.”
Solution: $\forall x \forall y (S(x,y) \rightarrow S(y,x))$

Example 3: “Everybody loves somebody.”
Solution: $\forall x \exists y L(x,y)$

Example 4: “There is someone who is loved by everyone.”
Solution: $\exists y \forall x L(x,y)$

Example 5: “There is someone who loves someone.”
Solution: $\exists x \exists y L(x,y)$

Example 6: “Everyone loves himself”
Solution: $\forall x L(x,x)$
Example 1: Recall the logical expression developed three slides back:
\[ \exists w \forall a \exists f \left( P(w,f) \land Q(f,a) \right) \]

Part 1: Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Solution: \[ \neg \exists w \forall a \exists f \left( P(w,f) \land Q(f,a) \right) \]

Part 2: Now use De Morgan’s Laws to move the negation as far inwards as possible.

Solution:
1. \[ \neg \exists w \forall a \exists f \left( P(w,f) \land Q(f,a) \right) \]
2. \[ \forall w \neg \forall a \exists f \left( P(w,f) \land Q(f,a) \right) \] by De Morgan’s for \( \exists \)
3. \[ \forall w \exists a \neg \exists f \left( P(w,f) \land Q(f,a) \right) \] by De Morgan’s for \( \forall \)
4. \[ \forall w \exists a \forall f \neg \left( P(w,f) \land Q(f,a) \right) \] by De Morgan’s for \( \exists \)
5. \[ \forall w \exists a \forall f \left( \neg P(w,f) \lor \neg Q(f,a) \right) \] by De Morgan’s for \( \land \).

Part 3: Can you translate the result back into English?

Solution:
“For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline”