Set Operations

Section 2.2
**Definition:** Let $A$ and $B$ be sets. The *union* of the sets $A$ and $B$, denoted by $A \cup B$, is the set:

$$\{ x | x \in A \lor x \in B \}$$

**Example:** What is $\{1,2,3\} \cup \{3,4,5\}$?

**Solution:** $\{1,2,3,4,5\}$
**Definition:** The *intersection* of sets $A$ and $B$, denoted by $A \cap B$, is

$$\{x | x \in A \land x \in B\}$$

- Note if the intersection is empty, then $A$ and $B$ are said to be *disjoint*.

- **Example:** What is? $\{1,2,3\} \cap \{3,4,5\}$?

  **Solution:** $\{3\}$

- **Example:** What is? $\{1,2,3\} \cap \{4,5,6\}$?

  **Solution:** $\emptyset$
Complement

**Definition:** If $A$ is a set, then the complement of the $A$ (with respect to $U$), denoted by $\bar{A}$ is the set $U - A$

$$\bar{A} = \{x \in U | x \notin A\}$$

(The complement of $A$ is sometimes denoted by $A^c$.)

**Example:** If $U$ is the positive integers less than 100, what is the complement of $\{x | x > 70\}$

**Solution:** $\{x | x \leq 70\}$
**Definition**: Let $A$ and $B$ be sets. The *difference* of $A$ and $B$, denoted by $A - B$, is the set containing the elements of $A$ that are not in $B$. The difference of $A$ and $B$ is also called the complement of $B$ with respect to $A$.

$$A - B = \{ x \mid x \in A \land x \notin B \} = A \cap \overline{B}$$

Venn Diagram for $A - B$
The Cardinality of the Union of Two Sets

- **Inclusion-Exclusion**
  \[ |A \cup B| = |A| + |B| - |A \cap B| \]

**Example**: Let \( A \) be the math majors in your class and \( B \) be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
Example: \( U = \{0,1,2,3,4,5,6,7,8,9,10\} \) \( A = \{1,2,3,4,5\} \), \( B = \{4,5,6,7,8\} \)

1. \( A \cup B \)
   Solution: \( \{1,2,3,4,5,6,7,8\} \)

2. \( A \cap B \)
   Solution: \( \{4,5\} \)

3. \( \overline{A} \)
   Solution: \( \{0,6,7,8,9,10\} \)

4. \( \overline{B} \)
   Solution: \( \{0,1,2,3,9,10\} \)

5. \( A - B \)
   Solution: \( \{1,2,3\} \)

6. \( B - A \)
   Solution: \( \{6,7,8\} \)
Symmetric Difference (optional)

**Definition:** The symmetric difference of $A$ and $B$, denoted by $A \oplus B$, is the set

$$A \oplus B$$

**Example:**

$$(A - B) \cup (B - A)$$

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$  $B = \{4,5,6,7,8\}$

What is:

- **Solution:** $\{1,2,3,6,7,8\}$
Set Identities

- Identity laws
  \[ A \cup \emptyset = A \quad A \cap U = A \]

- Domination laws
  \[ A \cup U = U \quad A \cap \emptyset = \emptyset \]

- Idempotent laws
  \[ A \cup A = A \quad A \cap A = A \]

- Complementation law
  \[ (\overline{A}) = A \]

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Set Identities

- Commutative laws
  \[ A \cup B = B \cup A \quad A \cap B = B \cap A \]

- Associative laws
  \[ A \cup (B \cup C) = (A \cup B) \cup C \]
  \[ A \cap (B \cap C) = (A \cap B) \cap C \]

- Distributive laws
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

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Set Identities

- **De Morgan’s laws**
  \[
  \overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}
  \]

- **Absorption laws**
  \[
  A \cup (A \cap B) = A \quad A \cap (A \cup B) = A
  \]

- **Complement laws**
  \[
  A \cup \overline{A} = U \quad A \cap \overline{A} = \emptyset
  \]
Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.
Example: Prove that \( A \cap B = \overline{A \cup B} \)

Solution: We prove this identity by showing that:

1) \( A \cap B \subseteq \overline{A \cup B} \) and

2) \( \overline{A \cup B} \subseteq A \cap B \)

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Proof of Second De Morgan Law

These steps show that: \( A \cap B \subseteq \overline{A} \cup \overline{B} \)

\[
\begin{align*}
x & \in \overline{A} \cap B \\
x & \notin A \cap B \\
\neg((x \in A) \land (x \in B)) & \quad \text{by assumption} \\
\neg(x \in A) \lor \neg(x \in B) & \quad \text{defn. of intersection} \\
x & \notin A \lor x \notin B \\
x & \in \overline{A} \lor x \in \overline{B} \\
x & \in \overline{A} \cup \overline{B}
\end{align*}
\]

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Proof of Second De Morgan Law

These steps show that: $\overline{A \cup B} \subseteq \overline{A \cap B}$

- $x \in \overline{A \cup B}$ by assumption
- $(x \in \overline{A}) \lor (x \in \overline{B})$ by defn. of union
- $(x \notin A) \lor (x \notin B)$ by defn. of complement
- $\neg(x \in A) \lor \neg(x \in B)$ by defn. of negation
- $\neg((x \in A) \land (x \in B))$ by 1st De Morgan Law for Prop Logic
- $\neg(x \in A \cap B)$ by defn. of intersection
- $x \in \overline{A \cap B}$ by defn. of complement
Set-Builder Notation: Second De Morgan Law

\[ \overline{A \cap B} = \{ x \mid x \notin A \cap B \} \]
\[ = \{ x \mid \neg (x \in (A \cap B)) \} \]
\[ = \{ x \mid \neg (x \in A \land x \in B) \} \]
\[ = \{ x \mid \neg (x \in A) \lor \neg (x \in B) \} \] by 1st De Morgan law for Prop Logic
\[ = \{ x \mid x \notin A \lor x \notin B \} \]
\[ = \{ x \mid x \in \overline{A} \lor x \in \overline{B} \} \]
\[ = \overline{A \cup B} \] by meaning of notation
Example: Construct a membership table to show that the distributive law holds.

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Solution:

<table>
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<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>B \cap C</th>
<th>A \cup (B \cap C)</th>
<th>A \cup B</th>
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<th>(A \cup B) \cap (A \cup C)</th>
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Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

We define:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

These are well defined, since union and intersection are associative.

For $i = 1, 2, \ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{n, n + 1, n + 2, \ldots\} = A_n$$