Computational Geometry (intro)

- Is the branch of computer science that studies algorithms for solving geometric problems.
- Has applications in many fields, including
  - computer graphics
  - robotics,
  - VLSI design
  - computer aided design
  - statistics
- Deals with geometric objects such as points, line segments, polygons, etc.
- Some typical problems considered:
  - whether intersections occur in a set of lines.
  - finding vertices of a convex hull for points.
  - whether a line can be drawn separating two sets of points.
  - whether one point is visible from a second point, given some polygons that may block visibility.
  - optimal location of fire towers to view a region.
  - closest or most distant pair of points.
  - whether a point is inside or outside a polygon.

Cross products

- ***Line segments***
  - The convex combination of two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is any point $p_3 = (x_3, y_3)$ such that for some real number α with $0 \le \alpha \le 1$,

$$(x_3, y_3) = \alpha(x_1, y_1) + (1-\alpha)(x_2, y_2).$$

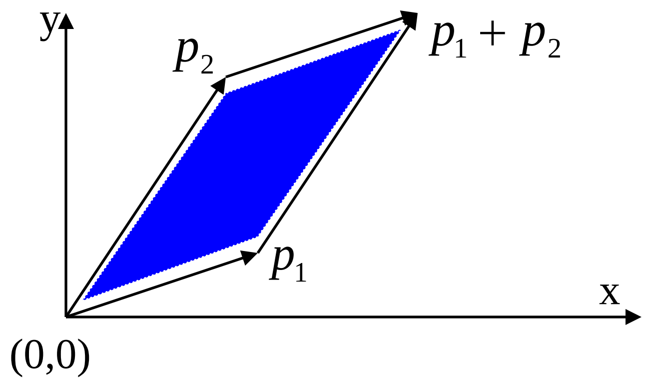
  - $\overline{p_1 p_2}$, the line segment joining p_1 and p_2, is the set of all convex combinations of p_1 and p_2.
  - *Intuition problem*: Show that if (x,y) is a convex combination of (x_1, y_1) and (x_2, y_2) then

$$\alpha = \frac{y - y_2}{x - x_2} = \frac{y_1 - y_2}{x_1 - x_2}$$

which is the standard equation of a line with slope α.

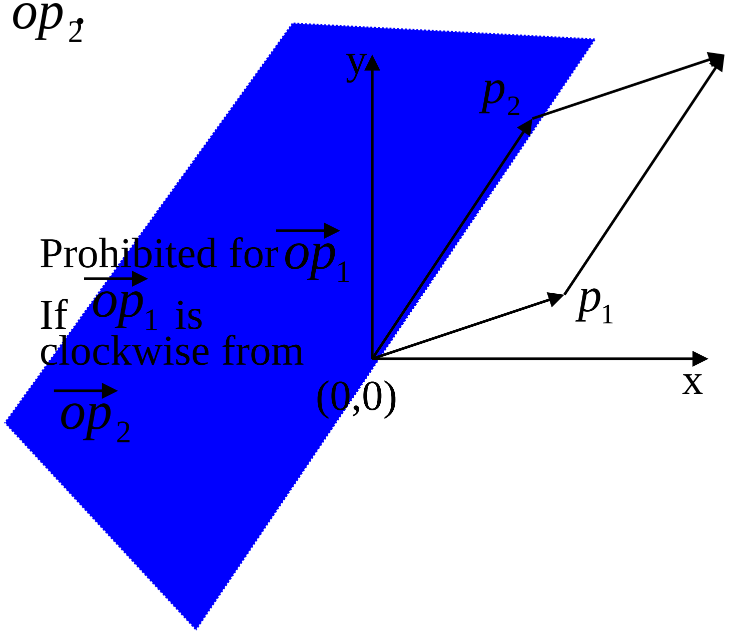
- ***Cross products***
  - let p_1 and p_2 be points on the plane
  - The cross product $p_1 \times p_2$ corresponds to the signed area in the parallelogram.

$$p_1 \times p_2 = \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 = -p_2 \times p_1.$$



Cross products (cont.)

- If $p_1 \times p_2$ is negative, then $\overrightarrow{op_1}$ is counterclockwise from $\overrightarrow{op_2}$.
- If $p_1 \times p_2$ is positive, then $\overrightarrow{op_1}$ is clockwise from $\overrightarrow{op_2}$.
- If $p_1 \times p_2 = 0,$ then $\overrightarrow{op_1}$ and $\overrightarrow{op_2}$ are collinear.

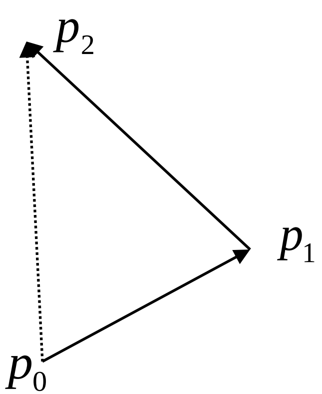


- To determine if $\overrightarrow{p_0 p_1}$ is clockwise from $\overrightarrow{p_0 p_2}$,

we translate p_0 to the origin and consider $p'_1 \times p'_2$ where $p'_1 = p_1 - p_0,\ p'_2 = p_2 - p_0.$

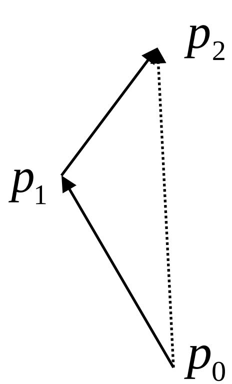
$$p'_1 \times p'_2 = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0).$$

- Consider now whether two consecutive line segments $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_1 p_2}$ turn ***left*** or ***right*** at p_1 .
  - Check whether $\overrightarrow{p_0 p_2}$ is clockwise from $\overrightarrow{p_0 p_1}$



$(p_2 - p_0) \times (p_1 - p_0) < 0$

So, counterclockwise or ***left turn***



$(p_2 - p_0) \times (p_1 - p_0) > 0$

So, clockwise or ***right turn***

Intersection of two line segments

- A two stage process is used. The first stage is called ***quick-rejection.***

- ***Bounding Box:*** The bounding box of a geometric figure is the smallest rectangle that contains the figure and whose segments are parallel to *x* and *y* axis.

- The bounding box $(\hat{p}_1, \hat{p}_2)$ of the line segment $\overline{p_1 p_2}$ is the box with lower left-hand point $\hat{p}_1 = (\hat{x}_1, \hat{y}_1)$, where

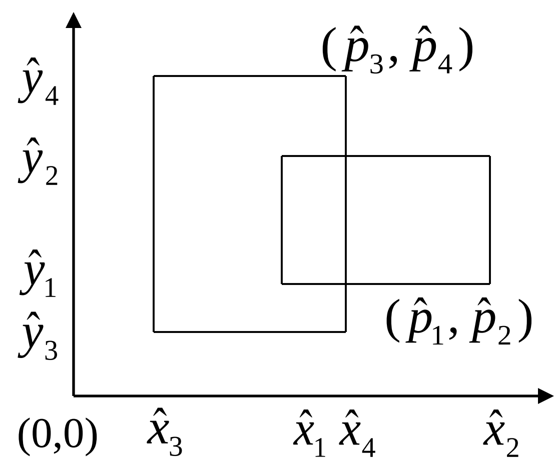
$$\hat{x}_1 = \min(x_1, x_2)$$
$$\hat{y}_1 = \min(y_1, y_2)$$

and with upper right-hand point $\hat{p}_2 = (\hat{x}_2, \hat{y}_2)$, where

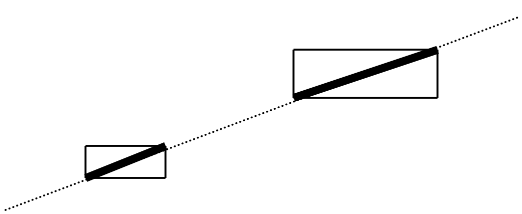
$$\hat{x}_2 = \max(x_1, x_2)$$
$$\hat{y}_2 = \max(y_1, y_2)$$

- Intersection of bounding boxes: Rectangles $(\hat{p}_1, \hat{p}_2)$ and $(\hat{p}_3, \hat{p}_4)$ intersect if and only if the following is true:

$$(\hat{x}_2 \geq \hat{x}_3) \Lambda (\hat{x}_4 \geq \hat{x}_1) \Lambda (\hat{y}_2 \geq \hat{y}_3) \Lambda (\hat{y}_4 \geq \hat{y}_1)$$



- this will not pass the quick rejection test

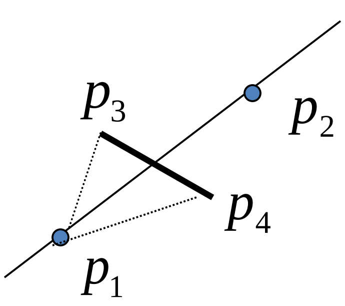


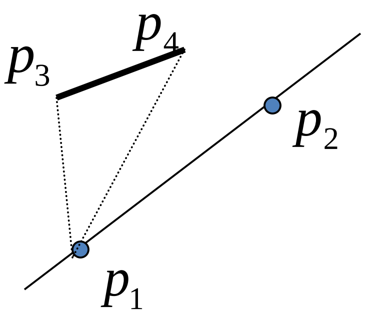
Intersection of two line segments (cont.)

- Second stage: Decide whether each segment meets ("straddles") the line containing the other.
- A segment $\overline{p_1 p_2}$ straddles a line if p_1 lies on one side of the line and p_2 on the other side. (the segment straddles the line also if p_1 or p_2 lies on the line)
- ***Observation:*** Two segments intersect iff they pass the quick rejection test and each segment straddles the other.
- Testing straddle with cross products:
  - we show how to check if $\overline{p_3 p_4}$ straddles the line L determined by p_1 and p_2

If $\overline{p_3 p_4}$ does straddle the line containing p_1 and p_2, then the following have different signs.

$$(p_3 - p_1) \times (p_2 - p_1)$$
$$(p_4 - p_1) \times (p_2 - p_1)$$





Boundary cases where $\overline{p_3 p_4}$ straddles L

At least one cross product is zero. Both cases pass the quick rejection test.