Fast Approximation of Centrality and Distances in Hyperbolic Graphs

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Abstract. We show that the eccentricities (and thus the centrality indices) of all vertices of a δ-hyperbolic graph $G = (V, E)$ can be computed in linear time with an additive one-sided error of at most $c\delta$, i.e., after a linear time preprocessing, for every vertex v of G one can compute in $O(1)$ time an estimate $\hat{e}(v)$ of its eccentricity $ecc_G(v)$ such that $ecc_G(v) \leq \hat{e}(v) \leq ecc_G(v)+c\delta$ for a small constant c. We prove that every δ-hyperbolic graph G has a shortest path tree, constructible in linear time, such that for every vertex v of G, $ecc_G(v) \leq ecc_T(v) \leq ecc_G(v)+c\delta$. We also show that the distance matrix of G with an additive one-sided error of at most $c'\delta$ can be computed in $O(|V|^2 \log^2 |V|)$ time, where $c' < c$ is a small constant. Recent empirical studies show that many real-world graphs (including Internet application networks, web networks, collaboration networks, social networks, biological networks, and others) have small hyperbolicity.

1 Introduction

The *diameter* $diam(G)$ and the *radius* $rad(G)$ of a graph $G = (V, E)$ are two fundamental metric parameters that have many important practical applications in real world networks. The problem of finding the *center* $C(G)$ of a graph G is often studied as a facility location problem for networks where one needs to select a single vertex to place a facility so that the maximum distance from any demand vertex in the network is minimized. In the analysis of social networks (e.g., citation networks or recommendation networks), biological systems (e.g., protein interaction networks), computer networks (e.g., the Internet or peer-to-peer networks), transportation networks (e.g., public transportation or road networks), etc., the *eccentricity* $ecc(v)$ of a vertex v is used to measure the importance of v in the network: the *centrality index* of v is defined as $\frac{1}{ecc(v)}$.

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Being able to compute efficiently the diameter, center, radius, and vertex centralities of a given graph has become an increasingly important problem in the analysis of large networks. The algorithmic complexity of the diameter and radius problems is very well-studied. For some special classes of graphs there are efficient algorithms [1,7,12,16,29]. However, for general graphs, the only known algorithms computing the diameter and the radius exactly compute the distance between every pair of vertices in the graph, thus solving the all-pairs shortest paths problem (APSP) and hence computing all eccentricities. In view of recent negative results [1,6,36], this seems to be the best what one can do since even for graphs with $m = O(n)$ (where m is the number of edges and n is the number of vertices) the existence of a subquadratic time (that is, $O(n^{2-\epsilon})$ time for some $\epsilon > 0$) algorithm for the diameter or the radius problem will refute the well known Strong Exponential Time Hypothesis (SETH). Furthermore, recent work [2] shows that if the radius of a possibly dense graph ($m = O(n^2)$) can be computed in subcubic time ($O(n^{3-\epsilon})$ for some $\epsilon > 0$), then APSP also admits a subcubic algorithm. Such an algorithm for APSP has long eluded researchers, and it is often conjectured that it does not exist.

Motivated by these negative results, researches started devoting more attention to development of fast approximation algorithms. In the analysis of large-scale networks, for fast estimations of diameter, center, radius, and centrality indices, linear or almost linear time algorithms are desirable. One hopes also for the all-pairs shortest paths problem to have $o(nm)$ time small-constant–factor approximation algorithms. In general graphs, both diameter and radius can be 2-approximated by a simple linear time algorithm which picks any node and reports its eccentricity. A 3/2-approximation algorithm for the diameter and the radius which runs in $\tilde{O}(mn^{2/3})$ time was recently obtained in [10] (see also [4] for an earlier $\tilde{O}(n^2 + m\sqrt{n})$ time algorithm and [36] for a randomized $\tilde{O}(m\sqrt{n})$ time algorithm). For the sparse graphs, this is an $o(n^2)$ time approximation algorithm. Furthermore, under plausible assumptions, no $O(n^{2-\epsilon})$ time algorithm can exist that $(3/2-\epsilon')$-approximates (for $\epsilon, \epsilon' > 0$) the diameter [36] and the radius [1] in sparse graphs. Similar results are known also for all eccentricities: a 5/3-approximation to the eccentricities of all vertices can be computed in $\tilde{O}(m^{3/2})$ time [10] and, under plausible assumptions, no $O(n^{2-\epsilon})$ time algorithm can exist that $(5/3-\epsilon')$-approximates (for $\epsilon, \epsilon' > 0$) the eccentricities of all vertices in sparse graphs [1]. Better approximation algorithms are known for some special classes of graphs [13,19,24,25].

Approximability of APSP is also extensively investigated. An additive 2-approximation for APSP in unweighted undirected graphs (the graphs we consider in this paper) was presented in [20]. It runs in $\tilde{O}(\min\{n^{3/2}m^{1/2}, n^{7/3}\})$ time and hence improves the runtime of an earlier algorithm from [4]. In [5], an $\tilde{O}(n^2)$ time algorithm was designed which computes an approximation of all distances with a multiplicative error of 2 and an additive error of 1. Furthermore, [5] gives an $O(n^{2.24+o(1)}\epsilon^{-3}\log(n/\epsilon))$ time algorithm that computes an approximation of all distances with a multiplicative error of $(1+\epsilon)$ and an additive error of 2.

Better algorithms are known for some special classes of graphs (see [7,13,23] and papers cited therein).

The need for fast approximation algorithms for estimating diameters, radii, centrality indices, or all pairs shortest paths in large-scale complex networks dictates to look for geometric and topological properties of those networks and utilize them algorithmically. The classical relationships between the diameter, radius, and center of trees and folklore linear time algorithms for their computation is one of the departing points of this research. A result from 1869 by C. Jordan [31] asserts that the radius of a tree T is roughly equal to half of its diameter and the center is either the middle vertex or the middle edge of any diametral path. The diameter and a diametral pair of T can be computed (in linear time) by a simple but elegant procedure: pick any vertex x, find any vertex y furthest from x, and find once more a vertex z furthest from y; then return $\{y, z\}$ as a diametral pair. One computation of a furthest vertex is called an *FP scan*; hence the diameter of a tree can be computed via two FP scans. This *two FP scans* procedure can be extended to exact or approximate computation of the diameter and radius in many classes of tree-like graphs. For example, this approach was used to compute the radius and a central vertex of a chordal graph in linear time [12]. In this case, the center of G is still close to the middle of all (y, z)-shortest paths and $d_G(y, z)$ is not the diameter but is still its good approximation: $d(y, z) \geq diam(G) - 2$. Even better, the diameter of any chordal graph can be approximated in linear time with an additive error 1 [25]. But it turns out that the exact computation of diameters of chordal graphs is as difficult as the general diameter problem: it is even difficult to decide if the diameter of a split graph is 2 or 3.

The experience with chordal graphs shows that one have to abandon the hope of having fast exact algorithms, even for very simple (from metric point of view) graph-classes, and to search for fast algorithms approximating $diam(G), rad(G), C(G), ecc_G(v)$ with a small additive constant depending only of the coarse geometry of the graph. *Gromov hyperbolicity* or the *negative curvature* of a graph (and, more generally, of a metric space) is one such constant. A graph $G = (V, E)$ is δ*-hyperbolic* [9,27,28] if for any four vertices w, v, x, y of G, the two largest of the three distance sums $d(w, v) + d(x, y)$, $d(w, x) + d(v, y)$, $d(w, y) + d(v, x)$ differ by at most $2\delta \geq 0$. The *hyperbolicity* $\delta(G)$ of a graph G is the smallest number δ such that G is δ-hyperbolic. The hyperbolicity can be viewed as a local measure of how close a graph is metrically to a tree: the smaller the hyperbolicity is, the closer its metric is to a tree-metric (trees are 0-hyperbolic and chordal graphs are 1-hyperbolic).

Recent empirical studies showed that many real-world graphs (including Internet application networks, web networks, collaboration networks, social networks, biological networks, and others) are tree-like from a metric point of view [3] or have small hyperbolicity [33,37]. It has been suggested in [33], and recently formally proved in [17], that the property, observed in real-world networks, in which traffic between nodes tends to go through a relatively small core of the network, as if the shortest paths between them are curved inwards, is due to the hyperbolicity of the network. Small hyperbolicity in real-world graphs provides also many algorithmic advantages. Efficient approximate solutions are attainable for a number of optimization problems [13,14,17,18,26,38].

In [13] we initiated the investigation of diameter, center, and radius problems for δ-hyperbolic graphs and we showed that the existing approach for trees can be extended to this general framework. Namely, it is shown in [13] that if G is a δ-hyperbolic graph and $\{y, z\}$ is the pair returned after two FP scans, then $d(y, z) \geq diam(G) - 2\delta$, $diam(G) \geq 2rad(G) - 4\delta - 1$, $diam(C(G)) \leq 4\delta + 1$, and $C(G)$ is contained in a small ball centered at a middle vertex of any shortest (y, z)-path. Consequently, we obtained linear time algorithms for the diameter and radius problems with additive errors linearly depending on the input graph's hyperbolicity. In this paper, we advance this line of research and provide a linear time algorithm for approximate computation of the eccentricities (and thus of centrality indices) of all vertices of a δ-hyperbolic graph G, i.e., we compute the approximate values of *all eccentricities* within the same time bounds as one computes the approximation of *the largest* or *the smallest eccentricity* ($diam(G)$ or $rad(G)$). Namely, the algorithm outputs for every vertex v of G an estimate $\hat{e}(v)$ of $ecc_G(v)$ such that $ecc_G(v) \leq \hat{e}(v) \leq ecc_G(v) + c\delta$, where $c > 0$ is a small constant. In fact, we demonstrate that G has a shortest path tree, constructible in linear time, such that for every vertex v of G, $ecc_G(v) \leq ecc_T(v) \leq ecc_G(v) + c\delta$ (a so-called *eccentricity $c\delta$-approximating spanning tree*). This is our first main result of this paper and the main ingredient in proving it is the following interesting dependency between the eccentricities of vertices of G and their distances to the center $C(G)$: up to an additive error linearly depending on δ, $ecc_G(v)$ is equal to $d(v, C(G))$ plus $rad(G)$. To establish this new result, we have to revisit the results of [13] about diameters, radii, and centers, by simplifying their proofs and extending them to all eccentricities.

Eccentricity k-approximating spanning trees were introduced by Prisner in [35]. A spanning tree T of a graph G is called an *eccentricity k-approximating spanning tree* if for every vertex v of G $ecc_T(v) \leq ecc_G(v) + k$ holds [35]. Prisner observed that any graph admitting an additive tree k-spanner (that is, a spanning tree T such that $d_T(v, u) \leq d_G(v, u) + k$ for every pair u, v) admits also an eccentricity k-approximating spanning tree. Therefore, eccentricity k-approximating spanning trees exist in interval graphs for $k = 2$ [32,34], in asteroidal-triple–free graph [32], strongly chordal graphs [8] and dually chordal graphs [8] for $k = 3$. On the other hand, although for every k there is a chordal graph without an additive tree k-spanner [32,34], yet as Prisner demonstrated in [35], every chordal graph has an eccentricity 2-approximating spanning tree. Later this result was extended in [24] to a larger family of graphs which includes all chordal graphs and all plane triangulations with inner vertices of degree at least 7. Both those classes belong to the class of 1-hyperbolic graphs. Thus, our result extends the result of [35] to all δ-hyperbolic graphs.

As our second main result, we show that in every δ-hyperbolic graph G all distances with an additive one-sided error of at most $c'\delta$ can be found in $O(|V|^2 \log^2 |V|)$ time, where $c' < c$ is a small constant. With a recent result in [11], this demonstrates an equivalence between approximating the hyperbolicity and approximating the distances in graphs. Note that every δ-hyperbolic graph G admits a distance approximating tree T [13,14], that is, a tree T (which is not necessarily a spanning tree) such that $d_T(v,u) \leq d_G(v,u) + O(\delta \log n)$ for every pair u, v. Such a tree can be used to compute all distances in G with an additive one-sided error of at most $O(\delta \log n)$ in $O(|V|^2)$ time. Our new result removes the dependency of the additive error from $\log n$ and has a much smaller constant in front of δ. Note also that tree T is not a spanning tree of G and thus cannot serve as an eccentricity $O(\delta \log n)$-approximating spanning tree. Furthermore, as chordal graphs are 1-hyperbolic, for every k there is a 1-hyperbolic graph without an additive tree k-spanner [32,34].

Finally, in the full version of the paper [15], we analyze the performance of our algorithms for approximating eccentricities and distances on a number of real-world networks. Our experimental results show that the estimates on eccentricities and distances obtained are even better than the theoretical bounds proved. Experimental results can be found in the full version of the paper [15].

2 Preliminaries

Center, Diameter, Centrality. All graphs $G = (V, E)$ occurring in this paper are finite, undirected, connected, without loops or multiple edges. We use n and $|V|$ interchangeably to denote the number of vertices and m and $|E|$ to denote the number of edges in G. The *length of a path* from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u,v)$ between vertices u and v is the length of a shortest path connecting u and v in G. The *eccentricity* of a vertex v, denoted by $ecc_G(v)$, is the largest distance from v to any other vertex, i.e., $ecc_G(v) = \max_{u \in V} d_G(v,u)$. The *centrality index* of v is $\frac{1}{ecc_G(v)}$. The *radius* $rad(G)$ of a graph G is the minimum eccentricity of a vertex in G, i.e., $rad(G) = \min_{v \in V} ecc_G(v)$. The *diameter* $diam(G)$ of a graph G is the the maximum eccentricity of a vertex in G, i.e., $diam(G) = \max_{v \in V} ecc_G(v)$. The *center* $C(G) = \{c \in V : ecc_G(c) = rad(G)\}$ of a graph G is the set of vertices with minimum eccentricity.

Gromov Hyperbolicity and Thin Geodesic Triangles. Let (X, d) be a metric space. The *Gromov product* of $y, z \in X$ with respect to w is defined to be $(y|z)_w = \frac{1}{2}(d(y,w) + d(z,w) - d(y,z))$. A metric space (X, d) is said to be δ-*hyperbolic* [28] for $\delta \geq 0$ if $(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta$ for all $w, x, y, z \in X$. Equivalently, (X, d) is δ-hyperbolic if for any four points u, v, x, y of X, the two largest of the three distance sums $d(u,v) + d(x,y)$, $d(u,x) + d(v,y)$, $d(u,y) + d(v,x)$ differ by at most $2\delta \geq 0$. A connected graph $G = (V, E)$ is δ-*hyperbolic* (or of *hyperbolicity* δ) if the metric space (V, d_G) is δ-hyperbolic, where d_G is the standard shortest path metric defined on G.

δ-Hyperbolic graphs generalize k-chordal and bounded tree-length graphs: each k-chordal graph has the tree-length at most $\lfloor \frac{k}{2} \rfloor$ [21] and each tree-length λ graph has hyperbolicity at most λ [13]. A graph is k-*chordal* if its induced cycles are of length at most k, and it is of *tree-length* λ if it has a tree-decomposition into bags of diameter at most λ [21].

For geodesic metric spaces and graphs there exist several equivalent definitions of δ-hyperbolicity involving different but comparable values of δ [9,27,28]. *In this paper, we will use the definition via thin geodesic triangles.* Let (X, d) be a metric space. A *geodesic* joining two points x and y from X is a (continuous) map f from the segment $[a, b]$ of $\mathbb{R}^1$ of length $|a - b| = d(x, y)$ to X such that $f(a) = x, f(b) = y$, and $d(f(s), f(t)) = |s - t|$ for all $s, t \in [a, b]$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic. Every graph $G = (V, E)$ can be transformed into a geodesic space (X, d) by replacing every edge $e = uv$ by a segment $[u, v]$ of length 1; the segments may intersect only at common ends. Then (V, d_G) is isometrically embedded in a natural way in (X, d). The restrictions of geodesics of X to the vertices V of G are the shortest paths of G.

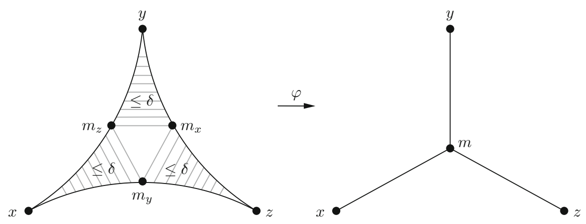

Fig. 1. A geodesic triangle $\Delta(x, y, z)$, the points m_x, m_y, m_z, and the tripod $\Upsilon(x, y, z)$

Let (X, d) be a geodesic metric space. A *geodesic triangle* $\Delta(x, y, z)$ with $x, y, z \in X$ is the union $[x, y] \cup [x, z] \cup [y, z]$ of three geodesic segments connecting these vertices. Let m_x be the point of the geodesic segment $[y, z]$ located at distance $\alpha_y := (x|z)_y = (d(y, x) + d(y, z) - d(x, z))/2$ from y. Then m_x is located at distance $\alpha_z := (y|x)_z = (d(z, y) + d(z, x) - d(y, x))/2$ from z because $\alpha_y + \alpha_z = d(y, z)$. Analogously, define the points $m_y \in [x, z]$ and $m_z \in [x, y]$ both located at distance $\alpha_x := (y|z)_x = (d(x, y) + d(x, z) - d(y, z))/2$ from x; see Fig. 1 for an illustration. There exists a unique isometry φ which maps $\Delta(x, y, z)$ to a tripod $T(x, y, z)$ consisting of three solid segments $[x, m], [y, m]$, and $[z, m]$ of lengths α_x, α_y, and α_z, respectively. This isometry maps the vertices x, y, z of $\Delta(x, y, z)$ to the respective leaves of $T(x, y, z)$ and the points m_x, m_y, and m_z to the center m of this tripod. Any other point of $T(x, y, z)$ is the image of exactly two points of $\Delta(x, y, z)$. A geodesic triangle $\Delta(x, y, z)$ is called δ-*thin* if for all points $u, v \in \Delta(x, y, z)$, $\varphi(u) = \varphi(v)$ implies $d(u, v) \le \delta$. A graph $G = (V, E)$ whose all geodesic triangles $\Delta(u, v, w)$, $u, v, w \in V$, are δ-thin is called a *graph with δ-thin triangles*, and δ is called the *thinness* parameter of G.

The following result shows that hyperbolicity of a geodesic space or a graph is equivalent to having thin geodesic triangles.

Proposition 1 **([9,27,28]).** *Geodesic triangles of geodesic δ-hyperbolic spaces or graphs are 4δ-thin. Conversely, geodesic spaces or graphs with δ-thin triangles are δ-hyperbolic.*

In what follows, we will need few more notions and notations. Let $G = (V, E)$ be a graph. By $[x, y]$ we denote a shortest path connecting vertices x and y in G; we call $[x, y]$ a *geodesic* between x and y. A *ball* $B(s, r)$ of G centered at vertex $s \in V$ and with radius r is the set of all vertices with distance no more than r from s (i.e., $B(s, r) := \{v \in V : d_G(v, s) \leq r\}$). The kth-*power* of a graph $G = (V, E)$ is the graph $G^k = (V, E')$ such that $xy \in E'$ if and only if $0 < d_G(x, y) \leq k$. Denote by $F(x) := \{y \in V : d_G(x, y) = ecc_G(x)\}$ the set of all vertices of G that are *most distant* from x. Vertices x and y of G are called *mutually distant* if $x \in F(y)$ and $y \in F(x)$, i.e., $ecc_G(x) = ecc_G(y) = d_G(x, y)$.

3 Fast Approximation of Eccentricities

In this section, we give linear and almost linear time algorithms for sharp estimation of the diameters, the radii, the centers and the eccentricities of all vertices in graphs with δ-thin triangles. Before presenting those algorithms, we establish some conditional lower bounds on complexities of computing the diameters and the radii in those graphs.

3.1 Conditional Lower Bounds on Complexities

Recent work has revealed convincing evidence that solving the diameter problem in subquadratic time might not be possible, even in very special classes of graphs. Roditty and Vassilevska W. [36] showed that an algorithm that can distinguish between diameter 2 and 3 in a sparse graph in subquadratic time refutes the following widely believed conjecture.

The Orthogonal Vectors Conjecture: There is no $\epsilon > 0$ such that for all $c \geq 1$, there is an algorithm that given two lists of n binary vectors $A, B \subseteq \{0, 1\}^d$ where $d = c \log n$ can determine if there is an orthogonal pair $a \in A, b \in B$, in $O(n^{2-e})$ time.

Williams [39] showed that the Orthogonal Vectors (OV) Conjecture is implied by the well-known Strong Exponential Time Hypothesis (SETH) of Impagliazzo, Paturi, and Zane [30]. Nowadays many papers base the hardness of problems on SETH and the OV conjecture (see, e.g., [1,6] and papers cited therein). Since all geodesic triangles of a graph constructed in the reduction in [36] are 2-thin, we can rephrase the result from [36] as follows.

Statement 1. *If for some $\epsilon > 0$, there is an algorithm that can determine if a given graph with 2-thin triangles, n vertices and $m = O(n)$ edges has diameter 2 or 3 in $O(n^{2-\epsilon})$ time, then the Orthogonal Vector Conjecture is false.*

To prove a similar lower bound result for the radius problem, recently Abboud et al. [1] suggested to use the following natural and plausible variant of the OV conjecture.

The Hitting Set Conjecture: There is no $\epsilon > 0$ such that for all $c \geq 1$, there is an algorithm that given two lists A, B of n subsets of a universe U of size $c \log n$, can decide in $O(n^{2-e})$ time if there is a set in the first list that intersects every set in the second list.

Abboud et al. [1] showed that an algorithm that can distinguish between radius 2 and 3 in a sparse graph in subquadratic time refutes the Hitting Set Conjecture. Since all geodesic triangles of a graph constructed in [1] are 2-thin, rephrasing that result from [1], we have.

Statement 2. *If for some $\epsilon > 0$, there is an algorithm that can determine if a given graph with 2-thin triangles, n vertices, and $m = O(n)$ edges has radius 2 or 3 in $O(n^{2-\epsilon})$ time, then the Hitting Set Conjecture is false.*

3.2 Fast Additive Approximations

In this subsection, we show that in a graph G with δ-thin triangles the eccentricities of all vertices can be computed in total linear time with an additive error depending on δ. We establish that the eccentricity of a vertex is determined (up-to a small error) by how far the vertex is from the center $C(G)$ of G. Finally, we show how to construct a spanning tree T of G in which the eccentricity of any vertex is its eccentricity in G up to an additive error depending only on δ. For these purposes, we revisit and extend several results from our previous paper [13] about diameters, radii, and centers of δ-hyperbolic graphs.

Define the eccentricity layers of a graph G as follows: for $k = 0, \ldots, diam(G) - rad(G)$ set $C^k(G) := \{v \in V : ecc_G(v) = rad(G) + k\}$. With this notation, the center of a graph is $C(G) = C^0(G)$. In what follows, it will be convenient to define also the eccentricity of the middle point m of any edge xy of G; set $ecc_G(m) = \min\{ecc_G(x), ecc_G(y)\} + 1/2$.

We start with a proposition showing that, in a graph G with δ-thin triangles, a middle vertex of any geodesic between two mutually distant vertices has the eccentricity close to $rad(G)$ and is not too far from the center $C(G)$ of G.

Proposition 2. *Let G be a graph with δ-thin triangles and u, v be a pair of mutually distant vertices of G.*

(a) If c^* is the middle point of any (u, v)-geodesic, then $ecc_G(c^*) \leq \frac{d_G(u,v)}{2} + \delta \leq rad(G) + \delta$.
(b) If c is a middle vertex of any (u, v)-geodesic, then $ecc_G(c) \leq \lceil \frac{d_G(u,v)}{2} \rceil + \delta \leq rad(G) + \delta$.
(c) $d_G(u, v) \geq 2rad(G) - 2\delta - 1$. In particular, $diam(G) \geq 2rad(G) - 2\delta - 1$.
(d) If c is a middle vertex of any (u, v)-geodesic and $x \in C^k(G)$, then $k - \delta \leq d_G(x, c) \leq k + 2\delta + 1$. In particular, $C(G) \subseteq B(c, 2\delta + 1)$.

Proof. Let x be any vertex of G and $\Delta(u, v, x) := [u, v] \cup [v, x] \cup [x, u]$ be a geodesic triangle, where $[x, v], [x, u]$ are arbitrary geodesics connecting x with v, u. Let m_x be a point on $[u, v]$ at distance $(x|u)_v = \frac{1}{2}(d(x, v) + d(v, u) - d(x, u))$ from v and at distance $(x|v)_u = \frac{1}{2}(d(x, u) + d(v, u) - d(x, v))$ from u. Since u and

v are mutually distant, we can assume that c^* is located on $[u,v]$ between v and m_x, i.e., $d(v,c^*) \leq d(v,m_x) = (x|u)_v$, and hence $(x|v)_u \leq (x|u)_v$. Since $d_G(v,x) \leq d_G(v,u)$, we also get $(u|v)_x \leq (x|v)_u$.

(a) By the triangle inequality and since $d_G(u,v) \leq diam(G) \leq 2rad(G)$, we get $d_G(x,c^*) \leq (u|v)_x + \delta + d_G(u,c^*) - (x|v)_u \leq d_G(u,c^*) + \delta = \frac{d_G(u,v)}{2} + \delta \leq rad(G) + \delta$.
(b) Since $c^* = c$ when $d_G(u,v)$ is even and $d_G(c^*,c) = \frac{1}{2}$ when $d_G(u,v)$ is odd, we have $ecc_G(c) \leq ecc_G(c^*) + \frac{1}{2}$. Additionally to the proof of (a), one needs only to consider the case when $d_G(u,v)$ is odd. We know that the middle point c^* sees all vertices of G within distance at most $\frac{d_G(u,v)}{2} + \delta$. Hence, both ends of the edge of (u,v)-geodesic, containing the point c^*, have eccentricities at most $\frac{d_G(u,v)}{2} + \frac{1}{2} + \delta = \lceil \frac{d_G(u,v)}{2} \rceil + \delta \leq \lceil \frac{2rad(G)-1}{2} \rceil + \delta = rad(G) + \delta$.
(c) Since a middle vertex c of any (u,v)-geodesic sees all vertices of G within distance at most $\lceil \frac{d_G(u,v)}{2} \rceil + \delta$, if $d_G(u,v) \leq 2rad(G) - 2\delta - 2$, then $ecc_G(c) \leq \lceil \frac{d_G(u,v)}{2} \rceil + \delta \leq \lceil \frac{2rad(G)-2\delta-2}{2} \rceil + \delta < rad(G)$, which is impossible.
(d) In the proof of (a), instead of any vertex x of G, consider any vertex x from $C^k(G)$. By the triangle inequality and since $d_G(u,v) \geq 2rad(G) - 2\delta - 1$ and both $d_G(u,x), d_G(x,v)$ are at most $rad(G) + k$, we get $d_G(x,c^*) \leq (u|v)_x + \delta + (x|u)_v - d_G(v,c^*) = d_G(v,x) - d_G(v,c^*) + \delta \leq rad(G) + k - \frac{d_G(u,v)}{2} + \delta \leq k + 2\delta + \frac{1}{2}$. Consequently, $d_G(x,c) \leq d_G(x,c^*) + \frac{1}{2} \leq k + 2\delta + 1$. On the other hand, since $ecc_G(x) \leq ecc_G(c) + d_G(x,c)$ and $ecc_G(c) \leq rad(G) + \delta$, by (a) we get $d_G(x,c) \geq ecc_G(x) - ecc_G(c) = k + rad(G) - ecc_G(c) \geq k - \delta$. □

As an easy consequence of Proposition 2(d), we get that the eccentricity $ecc_G(x)$ of any vertex x is equal, up to an additive one-sided error of at most $4\delta + 2$, to $d_G(x, C(G)) + rad(G)$ (a proof can be found in the full version of this paper [15]).

Corollary 1. *For every vertex x of a graph G with δ-thin triangles, $d_G(x, C(G)) + rad(G) - 4\delta - 2 \leq ecc_G(x) \leq d_G(x, C(G)) + rad(G)$.*

It is interesting to note that the equality $ecc_G(x) = d_G(x, C(G)) + rad(G)$ holds for every vertex of a graph G if and only if the eccentricity function $ecc_G(\cdot)$ on G is unimodal (that is, every local minimum is a global minimum)[22]. A slightly weaker condition holds for all chordal graphs [24]: for every vertex x of a chordal graph G, $ecc_G(x) \geq d_G(x, C(G)) + rad(G) - 1$. Proofs of the following two propositions can be found in the full version of the paper [15].

Proposition 3. *Let G be a graph with δ-thin triangles and u, v be a pair of vertices of G such that $v \in F(u)$.*

(a) If w is a vertex of a (u,v)-geodesic at distance $rad(G)$ from v, then $ecc_G(w) \leq rad(G) + \delta$.
(b) For every pair of vertices $x, y \in V$, $\max\{d_G(v,x), d_G(v,y)\} \geq d_G(x,y) - 2\delta$.

(c) $ecc_G(v) \geq diam(G) - 2\delta \geq 2rad(G) - 4\delta - 1$.
(d) *If* $t \in F(v)$, *c is a vertex of a* (v,t)*-geodesic at distance* $\lceil \frac{d_G(v,t)}{2} \rceil$ *from t and* $x \in C^k(G)$, *then* $ecc_G(c) \leq rad(G) + 3\delta$ *and* $k - 3\delta \leq d_G(x,c) \leq k + 3\delta + 1$. *In particular,* $C(G) \subseteq B(c, 3\delta + 1)$.

Proposition 4. *For every graph G with* δ*-thin triangles,* $diam(C^k(G)) \leq 2k + 2\delta + 1$. *In particular,* $diam(C(G)) \leq 2\delta + 1$.

Diameter and Radius. For any graph $G = (V, E)$ and any vertex $u \in V$, a most distant from u vertex $v \in F(u)$ can be found in linear $(O(|E|))$ time by a *breadth-first-search* $BFS(u)$ started at u. A pair of mutually distant vertices of a connected graph $G = (V, E)$ with δ-thin triangles can be computed in $O(\delta|E|)$ total time as follows. By Proposition 3(c), if v is a most distant vertex from u and t is a most distant vertex from v, then $d(v,t) \geq \text{diam}(G) - 2\delta$. Hence, using at most $O(\delta)$ *breadth-first-searches*, one can generate a sequence of vertices $v := v_1, t := v_2, v_3, \ldots v_k$ with $k \leq 2\delta + 2$ such that each v_i is most distant from v_{i-1} (with, $v_0 = u$) and v_k, v_{k-1} are mutually distant vertices (the initial value $d(v,t) \geq \text{diam}(G) - 2\delta$ can be improved at most 2δ times). By Proposition 2 and Proposition 3, we get the following additive approximations for the radius and the diameter of a graph with δ-thin triangles.

Corollary 2. *Let* $G = (V, E)$ *be a graph with* δ*-thin triangles.*

1. *There is a linear* $(O(|E|))$ *time algorithm which finds in G a vertex c with eccentricity at most* $rad(G) + 3\delta$ *and a vertex v with eccentricity at least* $diam(G) - 2\delta$. *Furthermore,* $C(G) \subseteq B(c, 3\delta + 1)$ *holds.*
2. *There is an almost linear* $(O(\delta|E|))$ *time algorithm which finds in G a vertex c with eccentricity at most* $rad(G) + \delta$. *Furthermore,* $C(G) \subseteq B(c, 2\delta + 1)$ *holds.*

All Eccentricities. In what follows, we will show that all vertex eccentricities of a graph with δ-thin triangles can be also additively approximated in (almost) linear time. It will be convenient, for the middle point m of an edge e of G, to define a $BFS(m)$-tree of G; it is nothing else than a $BFS(e)$-tree of G rooted at edge e.

Proposition 5. *Let G be a graph with* δ*-thin triangles.*

(a) *If v is a most distant vertex from an arbitrary vertex u, t is a most distant vertex from v, c is a vertex of a* (v,t)*-geodesic at distance* $\lceil \frac{d_G(v,t)}{2} \rceil$ *from t and T is a* $BFS(c)$*-tree of G, then* $ecc_G(x) \leq ecc_T(x) \leq ecc_G(x) + 6\delta + 1$.
(b) *If* c^* *is the middle point of any* (u,v)*-geodesic between a pair u, v of mutually distant vertices of G and T is a* $BFS(c^*)$*-tree of G, then, for every vertex x of G,* $ecc_G(x) \leq ecc_T(x) \leq ecc_G(x) + 2\delta$.

Proof. (a) Let x be an arbitrary vertex of G and assume that $ecc_G(x) = rad(G) + k$ for some integer $k \geq 0$. We know from Proposition 3(d) that $ecc_G(c) \leq rad(G) + 3\delta$ and $d_G(c,x) \leq k + 3\delta + 1$. Since T is a $BFS(c)$-tree, $d_G(x,c) = d_T(x,c)$ and $ecc_G(c) = ecc_T(c)$. Consider a vertex y in G such that $d_T(x,y) = ecc_T(x)$. We have $ecc_T(x) = d_T(x,y) \leq d_T(x,c) + d_T(c,y) \leq d_G(x,c) + ecc_T(c) = d_G(x,c) + ecc_G(c) \leq k + 3\delta + 1 + rad(G) + 3\delta = rad(G) + k + 6\delta + 1 = ecc_G(x) + 6\delta + 1$. As T is a spanning tree of G, evidently, also $ecc_G(x) \leq ecc_T(x)$ holds.

(b) Consider an arbitrary vertex x of G and a geodesic triangle $\Delta(x,u,v) := [x,u] \cup [u,v] \cup [v,x]$, where $[u,v]$ is a (u,v)-geodesic containing c^* and $[u,x],[v,x]$ are arbitrary geodesics connecting x with u and v. Let m_x be a point on $[u,v]$ which is at distance $(x|u)_v = \frac{1}{2}(d_G(x,v) + d_G(u,v) - d_G(x,u))$ from v and hence at distance $(x|v)_u = \frac{1}{2}(d_G(x,u) + d_G(v,u) - d_G(x,v))$ from u. Without loss of generality, we can assume that c^* is located on $[u,v]$ between v and m_x. We have, $d_G(x,c^*) \leq (u|v)_x + \delta + d_G(m_x,c^*) = (u|v)_x + \delta + d_G(u,c^*) - (v|x)_u = (u|v)_x + \delta + \frac{d_G(v,u)}{2} - (v|x)_u$, and $ecc_G(x) \geq d_G(x,v) = (u|v)_x + (u|x)_v$. Furthermore, by Proposition 2(a), $ecc_G(c^*) \leq \frac{d_G(v,u)}{2} + \delta$. Hence, $ecc_T(x) - ecc_G(x) \leq d_T(x,c^*) + ecc_T(c^*) - ecc_G(x) = d_G(x,c^*) + ecc_G(c^*) - ecc_G(x) \leq (u|v)_x + \delta + \frac{d_G(v,u)}{2} - (v|x)_u + \frac{d_G(v,u)}{2} + \delta - (u|v)_x - (u|x)_v = 2\delta + d_G(v,u) - ((v|x)_u + (u|x)_v) = 2\delta$. □

Theorem 1. *Every graph $G = (V,E)$ with δ-thin triangles admits an eccentricity (2δ)-approximating spanning tree constructible in $O(\delta|E|)$ time and an eccentricity $(6\delta+1)$-approximating spanning tree constructible in $O(|E|)$ time.*

Theorem 1 generalizes recent results from [24,35] that chordal graphs and some of their generalizations admit eccentricity 2-approximating spanning trees.

Note that the eccentricities of all vertices in any tree $T = (V,U)$ can be computed in $O(|V|)$ total time. As we noticed already, for trees the following facts are true: (1) $C(T)$ consists of one or two adjacent vertices; (2) $C(T)$ and $rad(T)$ of T can be found in linear time; (3) For any $v \in V$, $ecc_T(v) = d_T(v,C(T)) + rad(T)$. Hence, using $BFS(C(T))$ on T one can compute $d_T(v,C(T))$ for all $v \in V$ in total $O(|V|)$ time. Adding now $rad(T)$ to $d_T(v,C(T))$, one gets $ecc_T(v)$ for all $v \in V$. Consequently, by Theorem 1, we get the following additive approximations for the vertex eccentricities in graphs with δ-thin triangles.

Theorem 2. *Let $G = (V,E)$ be a graph with δ-thin triangles.*

(1) There is an algorithm which in total linear $(O(|E|))$ time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $ecc_G(v)$ such that $ecc_G(v) \leq \hat{e}(v) \leq ecc_G(v) + 6\delta + 1$.

(2) There is an algorithm which in total almost linear $(O(\delta|E|))$ time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $ecc_G(v)$ such that $ecc_G(v) \leq \hat{e}(v) \leq ecc_G(v) + 2\delta$.

4 Fast Additive Approximation of All Distances

Here, we will show that if the δth power G^δ of a graph G with δ-thin triangles is known in advance, then the distances in G can be additively approximated in

$O(|V|^2)$ time. If G^δ is not known, then the distances can be additively approximated in almost quadratic time.

Our method is a generalization of an unified approach used in [23] to estimate (or compute exactly) all pairs shortest paths in such special graph families as k-chordal graphs, chordal graphs, AT-free graphs and many others. For example: all distances in k-chordal graphs with an additive one-sided error of at most $k-1$ can be found in $O(|V|^2)$ time; all distances in chordal graphs with an additive one-sided error of at most 1 can be found in $O(|V|^2)$ time and the all pairs shortest path problem on a chordal graph G can be solved in $O(|V|^2)$ time if G^2 is known. Note that in chordal graph all geodesic triangles are 2-thin.

Let $G = (V, E)$ be a graph with δ-thin triangles. Pick an arbitrary start vertex $s \in V$ and construct a *$BFS(s)$-tree T* of G *rooted at s*. Denote by $p_T(x)$ the *parent* and by $h_T(x) = d_T(x, s) = d_G(x, s)$ the *height* of a vertex x in T. Since we will deal only with one tree T, we will often omit the subscript T. Let $P_T(x, s) := (x_q, x_{q-1}, \ldots, x_1, s)$ and $P_T(y, s) := (y_p, y_{p-1}, \ldots, y_1, s)$ be the paths of T connecting vertices x and y with the root s. By $sl_T(x, y; \lambda)$ we denote the largest index k such that $d_G(x_k, y_k) \le \lambda$ (the λ separation level). Our method is based on the following simple fact.

Proposition 6. *For every vertices x and y of a graph G with δ-thin triangles and any BFS-tree T of G, $h_T(x)+h_T(y)-2k-1 \le d_G(x, y) \le h_T(x)+h_T(y)-2k+d_G(x_k, y_k)$, where $k = sl_T(x, y; \delta)$.*

Proof. By the triangle inequality, $d_G(x, y) \le d_G(x, x_k)+d_G(x_k, y_k)+d_G(y_k, y) = h_T(x)+h_T(y)-2k+d_G(x_k, y_k)$. Consider now an arbitrary (x, y)-geodesic $[x, y]$ in G. Let $\Delta(x, y, s) := [x, y] \cup [x, s] \cup [y, s]$ be a geodesic triangle, where $[x, s] = P_T(x, s)$ and $[y, s] = P_T(y, s)$. Since $\Delta(x, y, s)$ is δ-thin, $sl_T(x, y; \delta) \ge (x|y)_s - \frac{1}{2}$. Hence, $h_T(x) - sl_T(x, y; \delta) \le (s|y)_x + \frac{1}{2}$ and $h_T(y) - sl_T(x, y; \delta) \le (s|x)_y + \frac{1}{2}$. As $d_G(x, y) = (s|y)_x + (s|x)_y$, we get $d_G(x, y) \ge h_T(x) - sl_T(x, y; \delta) + h_T(y) - sl_T(x, y; \delta) - 1$. □

Note that we may regard $BFS(s)$ as having produced a numbering from n to 1 in decreasing order of the vertices in V where vertex s is numbered n. As a vertex is placed in the queue by $BFS(s)$, it is given the next available number. The last vertex visited is given the number 1. Let $\sigma := [v_1, v_2, \ldots, v_n = s]$ be a $BFS(s)$-ordering of the vertices of G and T be a $BFS(s)$-tree of G produced by a $BFS(s)$. Let $\sigma(x)$ be the number assigned to a vertex x in this $BFS(s)$-ordering. For two vertices x and y, we write $x < y$ whenever $\sigma(x) < \sigma(y)$.

First, we show that if G^δ is known in advance (i.e., its adjacency matrix is given) for a graph G with δ-thin triangles, then the distances in G can be additively approximated (with an additive one-sided error of at most $\delta + 1$) in $O(|V|^2)$ time. We consider the vertices of G in the order σ from 1 to n. For each current vertex x we show that the values $\widehat{d}(x, y) := h_T(x) + h_T(y) - 2sl_T(x, y; \delta)+\delta$ for all vertices y with $y > x$ can be computed in $O(|V|)$ total time. By Proposition 6, $d_G(x, y) \le \widehat{d}(x, y) \le d_G(x, y)+\delta+1$. The values $\widehat{d}(x, y)$ for all y with $y > x$ can be computed using the following simple procedure. We omit the subscripts G and T if no ambiguities arise. Let also $L_i = \{v \in V : d(v,s) = i\}$. In the procedure, S_u represents vertices of a subtree of T rooted at u.

(01) set $q := h(x)$
(02) let $S_u := \{u\}$ for each vertex $u \in L_q$, $u > x$, and denote this family of sets by $\mathcal{F}$
(03) **for** $k = q$ downto 0 **do**
(04) let x_k be the vertex from $L_k \cap P_T(x,s)$
(05) **for** each vertex $u \in L_k$ with $u > x$ **do**
(06) **if** $d_G(u,x_k) \le \delta$ (i.e., $u = x_k$ or u is adjacent to x_k in G^δ) **then**
(07) **for** every $v \in S_u$ **do**
(08) set $\widehat{d}(x,v) := h(x) + h(v) - 2k + \delta$ and remove S_u from $\mathcal{F}$
(09) /* update $\mathcal{F}$ for the next iteration */
(10) **if** $k > 0$ **then**
(11) **for** each vertex $u \in L_{k-1}$ **do**
(12) combine sets $S_{u_1}, \ldots, S_{u_\ell}$ from $\mathcal{F}$ ($\ell \ge 0$) with $p_T(u_1) = \ldots = p_T(u_\ell) = u$
(13) into one new set $S_u := \{u\} \cup S_{u_1} \cup \ldots \cup S_{u_\ell}$ /* when $\ell = 0$, $S_u := \{u\}$ */
(14) set also $\widehat{d}(x,s) := h(x)$.

Theorem 3. *Let $G = (V,E)$ be a graph with δ-thin triangles. Given G^δ, all distances in G with an additive one-sided error of at most $\delta + 1$ can be found in $O(|V|^2)$ time.*

To avoid the requirement that G^δ is given in advance, we can use any known fast constant-factor approximation algorithm that in total $T(|V|)$-time computes for every pair of vertices x, y of G a value $\widetilde{d}(x,y)$ such that $d_G(x,y) \le \widetilde{d}(x,y) \le \alpha d_G(x,y) + \beta$. We can show that, using such an algorithm as a preprocessing step, the distances in a graph G with δ-thin triangles can be additively approximated with an additive one-sided error of at most $\alpha\delta + \beta + 1$ in $O(T(|V|) + |V|^2)$ time. Although one can use any known fast constant-factor approximation algorithm in the preprocessing step, in what follows, we will demonstrate our idea using a fast approximation algorithm from [5]. It computes in $O(|V|^2 \log^2 |V|)$ total time for every pair x, y a value $\widetilde{d}(x,y)$ such that $d_G(x,y) \le \widetilde{d}(x,y) \le 2d_G(x,y) + 1$. Assume that the values $\widetilde{d}(x,y)$, $x, y \in V$, are precomputed. By $\widetilde{sl}_T(x,y;\lambda)$ we denote now the largest index k such that $\widetilde{d}_G(x_k,y_k) \le \lambda$. We have.

Proposition 7. *For every vertices x and y of a graph G with δ-thin triangles, any integer $\rho \ge \delta$, and any BFS-tree T of G, $h_T(x) + h_T(y) - 2k - 1 \le d_G(x,y) \le h_T(x) + h_T(y) - 2k + d_G(x_k,y_k)$, where $k = \widetilde{sl}_T(x,y;2\rho+1)$.*

Proof of this propositions can be found in the full version of the paper [15]. Let ρ be any integer greater than or equal to δ. By replacing in our earlier procedure lines (06) and (08) with

(06)′ **if** $\widetilde{d}(u,x_k) \le 2\rho + 1$ **then**
(08)′ set $\widehat{\widetilde{d}}(x,v) := h(x) + h(v) - 2k + 2\rho + 1$ and remove S_u from $\mathcal{F}$

we will compute for each current vertex x all values $\widehat{d}(x,y) := h_T(x) + h_T(y) - 2\widetilde{sl}_T(x,y;2\rho{+}1){+}2\rho{+}1$, $y > x$, in $O(|V|)$ total time. By Proposition 7, $d_G(x,y) \leq h_T(x) + h_T(y) - 2\widetilde{sl}_T(x,y;2\rho+1) + d_G(x_k,y_k) \leq h_T(x) + h_T(y) - 2\widetilde{sl}_T(x,y;2\rho+1) + \widetilde{d}(x_k,y_k) \leq h_T(x) + h_T(y) - 2\widetilde{sl}_T(x,y;2\rho+1) + 2\rho + 1 = \widehat{d}(x,y)$ and $\widehat{d}(x,y) = h_T(x) + h_T(y) - 2\widetilde{sl}_T(x,y;2\rho+1) + 2\rho + 1 \leq d_G(x,y) + 2\rho + 2$. Thus, we have the following result:

Theorem 4. *Let $G = (V,E)$ be a graph with δ-thin triangles.*

(a) If the value of δ is known, then all distances in G with an additive one-sided error of at most $2\delta + 2$ can be found in $O(|V|^2 \log^2 |V|)$ time.
(b) If an approximation ρ of δ such that $\delta \leq \rho \leq a\delta + b$ is known (where a and b are constants), then all distances in G with an additive one-sided error of at most $2(a\delta + b + 1)$ can be found in $O(|V|^2 \log^2 |V|)$ time.

The second part of Theorem 4 says that if an approximation of the thinness of a graph G is given, then all distances in G can be additively approximated in $O(|V|^2 \log^2 |V|)$ time. Recently, it was shown in [11] that the converse is also true. From an estimate of all distances in G with an additive one-sided error of at most k, it is possible to compute in $O(|V|^2)$ time an estimation ρ^* of the thinness of G such that $\delta \leq \rho^* \leq 8\delta + 12k + 4$, proving a $\tilde{O}(|V|^2)$-equivalence between approximating the thinness and approximating the distances in graphs.

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