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CONVEXITY AND HHD-FREE GRAPHS*

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Abstract. It is well known that chordal graphs can be characterized via m-convexity. In this paper we introduce the notion of m^3-convexity (a relaxation of m-convexity) which is closely related to semisimplicial orderings of graphs. We present new characterizations of HHD-free graphs via m^3-convexity and obtain some results known from [B. Jamison and S. Olariu, Adv. Appl. Math., 9 (1988), pp. 364–376] as corollaries. Moreover, we characterize weak bipolarizable graphs as the graphs for which the family of all m^3-convex sets is a convex geometry. As an application of our results we present a simple efficient criterion for deciding whether a HHD-free graph contains a r-dominating clique with respect to a given vertex radius function r.

Key words. convexity, convex geometry, antimatroid, chordal graphs, HHD-free graphs, weak bipolarizable graphs, semisimplicial ordering, lexicographic breadth first search, dominating clique problem

AMS subject classifications. 05C65, 05C75, 68R10

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1. Introduction.

This paper was inspired by the results of Farber and Jamison [16] on convexity in chordal graphs and by the results of Jamison and Olariu [19] on semisimplicial orderings of graphs produced by "lexicographic breadth first search" (LexBFS) [25] and "maximum cardinality search" (MCS) [28].

Throughout this paper all graphs $G = (V, E)$ are finite, undirected, and simple (i.e., loop-free and without multiple edges). The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set V, where two vertices are adjacent in $\overline{G}$ iff they are nonadjacent in G.

A *path* is a sequence of vertices $v_0, \ldots, v_l$ such that $v_iv_{i+1} \in E$ for $i = 0, \ldots, l-1$; its *length* is l. An *induced path* is a path, where $v_iv_j \in E$ iff $i = j-1$ and $j = 1, \ldots, l$. An *induced cycle* is a sequence of vertices $v_0, \ldots, v_k$ such that $v_0 = v_k$ and $v_iv_j \in E$ iff $|i-j| = 1$ (modulo k). The *length* $|C|$ of a cycle C is its number of vertices. Let also $|P|$ be the number of vertices of a path P. A *hole* is an induced cycle of length at least five, whereas an *antihole* is the complement of a hole. By P_k we denote an induced path on k vertices. A graph G is *connected* iff for any pair of vertices of G there is a path in G joining these vertices. A set $S \subset V$ is *connected* in G iff the subgraph $G(S)$ induced by S is connected.

The *distance* $d_G(u, v)$ between two vertices u, v is the minimum number of edges on a path connecting these vertices, and is infinite if u and v lie in distinct connected components of the graph G. If no confusion can arise we will omit the index G. For

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a vertex $v \in V$ and a set $S \subseteq V$ we denote by $d(v,S)$ the minimum over all distances $d(v,s)$, $s \in S$. A subgraph H of a graph G is *isometric* iff the distance between any pair of vertices in H is the same as that in G.

The kth *neighborhood* $N^k(v)$ of a vertex v of G is the set of all vertices of distance k to v, i.e.,

$$N^k(v) := \{u \in V : d_G(u,v) = k\},$$

whereas the *disk* of radius k centered at v is the set of all vertices of distance at most k to v:

$$D_G(v,k) := \{u \in V : d_G(u,v) \leq k\}.$$

Again, if no confusion arises we will omit the index G. We also write $N(v)$ instead of $N^1(v)$.

The *eccentricity* $e(v)$ of a vertex $v \in V$ is the maximum value of $d(v,x)$ taken over all vertices $x \in V$. The *radius* $rad(G)$ of G is the minimum eccentricity of a vertex of G, whereas the *diameter* $diam(G)$ of G is the maximum eccentricity of a vertex of G.

Now we will give a short introduction to the theory of convex geometry related to graph theory following [16] (for more information on abstract convexity and antimatroids the interested reader can consult [21]). Let V be a finite set and $\mathfrak{M}$ be a family of subsets of V. $\mathfrak{M}$ is called *alignment* of V iff the family $\mathfrak{M}$ is closed under intersection and contains both V and the empty set. Elements of $\mathfrak{M}$ will be considered as convex sets. An *aligned space* is a pair $(V, \mathfrak{M})$, where $\mathfrak{M}$ is an alignment of V.

The smallest member of $\mathfrak{M}$ containing a given set $S \subseteq V$ is the *hull* of S, denoted by $\mathfrak{M}(S)$. An element x of a set $X \in \mathfrak{M}$ is an *extreme point* of X iff $X \setminus \{x\} \in \mathfrak{M}$.

The Caratheodory number of an aligned space $(V, \mathfrak{M})$ is the minimum integer k such that for all $X \subseteq V$, $\mathfrak{M}(X)$ is the union of the hulls of all subsets Y of X such that $|Y| \leq k$.

A *convex geometry (antimatroid)* on a finite set is an aligned space satisfying the following additional property.

Minkowski–Krein–Milman property. Every convex set is the hull of its extreme points.

Equivalently, a convex geometry is an aligned space satisfying the following property.

Antiexchange property. For any convex set S and two distinct points $x, y \notin S$, $x \in \mathfrak{M}(S \cup \{y\})$ implies $y \notin \mathfrak{M}(S \cup \{x\})$.

For any convex geometry the following fundamental result holds.

THEOREM 1.1 (see [16]). *If $(V, \mathfrak{M})$ is a convex geometry, then $S \in \mathfrak{M}$ iff is an ordering $(x_1, \ldots, x_k)$ of $V \setminus S$ such that x_i is an extreme point of $S \cup \{x_i, \ldots, x_k\}$ for each $i = 1, \ldots, k$.*

For a given ordering $(v_1, \ldots, v_n)$ of the vertex set of a graph $G = (V,E)$ let $G_i := G(\{v_i, \ldots, v_n\})$ be the subgraph of G induced by the set $\{v_i, \ldots, v_n\}$, $i = 1, \ldots, n$.

Numerous classes of graphs can be characterized in the following way. G is a member of class $\mathfrak{G}$ iff there is an ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that v_i satisfies a certain property $\mathbf{P}$ in the subgraph G_i, $i = 1, \ldots, n$.

Theorem 1.1 suggests that such classes of graphs might be related to convex geometries, and so it is natural to ask for a graph theoretical description of convex sets of this aligned space. On the other hand, given a collection $\mathfrak{M}$ of subsets of $V(G)$, one can ask when $(V(G), \mathfrak{M})$ is a convex geometry.

For example, if property **P** means "is simplicial" then $\mathfrak{G}$ is the class of *chordal* graphs, i.e., the graphs without induced cycles of length at least four [7, 24]. A vertex v of G is called *simplicial* iff $D(v, 1)$ induces a complete subgraph of G, and *nonsimplicial* otherwise. It is well known that a graph is chordal iff it has a *perfect elimination ordering*, i.e., an ordering $(v_1, \ldots, v_n)$ of V such that v_i is simplicial in G_i for each $i = 1, \ldots, n$ (cf. [7, 24]). Moreover, there are two linear time algorithms for computing perfect elimination orderings of chordal graphs: LexBFS [25] and MCS [28].

Two types of convexity in graphs have been studied most extensively, namely, monophonic (m-) convexity and geodesic (g-) convexity (see, e.g., [4, 12, 13, 14, 15, 16, 17, 20, 22, 26, 27]). A set $S \subseteq V(G)$ is *m-convex* (*g-convex*) iff S contains every vertex on every induced (shortest) path between vertices in S. Both types of convexity have a relation to simplicial vertices; a vertex v is an extreme point of a m-convex (g-convex) set S iff v is simplicial in $G(S)$. In [16] it is shown that G is a chordal graph iff the monophonic alignment of G is a convex geometry, while the geodesic alignment of G is a convex geometry iff G is a chordal graph without induced 3-fan (i.e., a P_4 with an additional vertex adjacent to all vertices of P_4). To prove that the monophonic alignment of a chordal graph is a convex geometry, the authors of [16] show the following nice result. Every nonsimplicial vertex of a chordal graph lies on an induced path between simplicial vertices.

For any notion of convexity on the vertex set of G, at least four degrees of local convexity may be distinguished [17]:

(1.1) $D(v, 1)$ is convex for every vertex v of G,

(1.2) $D(v, k)$ is convex for every vertex v of G and every $k \geq 1$,

(1.3) $\bigcup_{v \in S} D(v, 1)$ is convex for every convex subset $S \subseteq V$ of G,

(1.4) $\bigcup_{v \in S} D(v, k)$ is convex for every convex subset $S \subseteq V$ of G and every $k \geq 1$.

In [16] it was shown that for m-convexity the conditions (1.1)–(1.4) are equivalent and hold iff the graph is chordal. For g-convexity conditions (1.1)–(1.3) are not equivalent (note that (1.3) implies (1.4) for any convexity in graphs [17]). Several characterizations for graphs with property (1.1), (1.2), or (1.3) are given in [14, 17, 27]. Here we will mention only one result which clearly shows an analogy with chordal graphs. Namely, a graph G fulfills the condition (1.3) iff G is a bridged graph, i.e., a graph which contains no isometric cycles of length at least four.

Note that a vertex is simplicial iff it is not midpoint of a P_3. Jamison and Olariu relaxed this condition in [19] in the following way: A vertex is *semisimplicial* iff it is not a midpoint of a P_4, and *nonsemisimplicial* otherwise. An ordering $(v_1, \ldots, v_n)$ is a *semisimplicial ordering* iff v_i is semisimplicial in G_i for all $i = 1, \ldots, n$. In [19] the authors characterized the graphs for which every LexBFS-ordering is a semisimplicial ordering as the HHD-free graphs, i.e., the graphs which contain no house, hole, or domino as an induced subgraph (cf. Figure 1). Moreover, the graphs for which every MCS-ordering of an arbitrary induced subgraph F is a semisimplicial ordering of F are the HHP-free graphs, i.e., the graphs which contain no house, hole, or "P" as an induced subgraph (cf. Figure 1).

If a HHD-free graph G does not contain the "A" of Figure 1 as an induced subgraph then G is called *weak bipolarizable* (HHDA-free) [23].

In this paper we introduce the notion of m^3-convexity (a relaxation of m-convexity), which is closely related to semisimpliciality. A subset $S \subseteq V$ is called *m^3-convex* iff

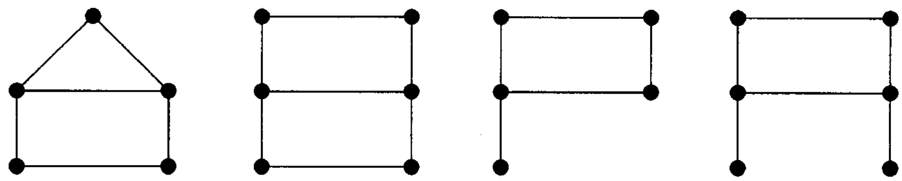

FIG. 1.

for any pair of vertices x, y of S each induced path of length at least 3 connecting x and y is completely contained in S. Note that a m^3-convex set is not necessarily connected, and it is not difficult to see that the family of m^3-convex sets is closed under intersection. Observe also that a vertex v is an extreme point of a m^3-convex set S iff v is semisimplicial in $G(S)$.

In this paper we present new characterizations of HHD-free and HHDA-free graphs via m^3-convexity. We show that for m^3-convexity the conditions (1.1)–(1.4) are again equivalent and hold iff the graph is HHD-free. We characterize weak bipolarizable graphs as the graphs for which the m^3-convex alignment is a convex geometry, i.e., by Theorem 1.1, for which every m^3-convex set is reachable via some semisimplicial ordering. Again, as for chordal graphs, in weak bipolarizable graphs every nonsemisimplicial vertex lies on an induced path of length at least 3 between semisimplicial vertices.

Convexity in graphs is a useful tool not only for geometric characterizations of several graph classes but also for resolving some problems related to distances in graphs [1, 4, 5, 6, 9, 14, 22]. As an application of our results we present a simple efficient criterion for deciding whether a HHD-free graph $G = (V, E)$ with given vertex radius function $r : V \to \mathbb{N}$ has an r-dominating clique. Note that this problem is $\mathbb{NP}$-complete for weakly chordal graphs (i.e., the graphs without holes and antiholes) [2]. From this criterion we obtain the inequality $diam(G) \geq 2rad(G) - 2$ between the diameter and radius of a HHD-free graph G. These results extend the known ones for chordal, distance-hereditary, and house-hole-domino-sun–free graphs [3, 5, 8, 9, 10].

Thus, the results of the paper show strict analogies between these graphs and chordal graphs. HHD-free, HHDA-free, and HHP-free graphs are three very natural generalizations of the class of chordal graphs.

2. m^3-convex sets in HHD-free graphs. In this section we characterize HHD-free graphs as the graphs with m^3-convex disks. Using m^3-convexity we give new properties of LexBFS-and MCS-orderings in HHD-free graphs and obtain known results from [19] as corollaries.

Since a vertex v is an extreme point of a m^3-convex set S iff v is semisimplicial in $G(S)$, we immediately conclude the following.

LEMMA 2.1. *An ordering $(v_1, \ldots, v_n)$ of the vertices of a graph G is semisimplicial iff $V(G_i) = \{v_i, \ldots, v_n\}$ is m^3-convex in G for all $i = 1, \ldots, n$.*

The following lemma will be frequently used in what follows.

LEMMA 2.2 (cycle lemma for hole-free graphs). *Let C be a cycle of length at least 5 in a hole-free graph G. Then for each edge xy of C there are vertices w_1, w_2 in C such that $xw_1 \in E$, $yw_2 \in E$, and $d(w_1, w_2) \leq 1$, i.e., each edge of a cycle is contained in a triangle or a 4-cycle.*

Proof. By induction on the length of the cycle. ∎

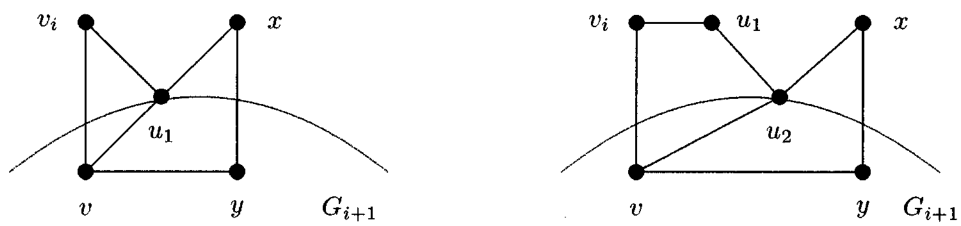

FIG. 2.

To make the paper self-contained we present the rules of the LexBFS and MCS algorithms.

LexBFS: Order vertices of a graph by assigning numbers from $n = |V|$ to 1. Assign the number k to a vertex v (as yet unnumbered), which has lexically largest vector $(s_i : i = n, n-1, \dots, k+1)$, where $s_i = 1$ if v is adjacent to the vertex numbered i, and $s_i = 0$ otherwise.

MCS: Order vertices of a graph by assigning numbers from $n = |V|$ to 1. As the next vertex to number pick a vertex adjacent to the most numbered vertices.

Subsequently, we will write $x < y$ whenever in a given ordering of the vertex set of a graph G vertex x has a smaller number than vertex y.

In what follows we will often use the following properties:

$(P1)$ If $a < b < c$ and $ac \in E$ and $bc \notin E$, then there exists a vertex d such that $c < d$, $db \in E$, and $da \notin E$.

$(P2)$ If $a < b < c$ and $ac \in E$ and $bc \notin E$, then there exists a vertex d such that $b < d$, $db \in E$, and $da \notin E$.

Evidently, $(P2)$ is a relaxation of $(P1)$. It is well known that any LexBFS-ordering has property $(P1)$ [18] and any MCS-ordering has property $(P2)$ [28].

THEOREM 2.3.

(1) *Let G be a HHD-free graph and $(v_1, \dots, v_n)$ be a LexBFS-ordering of G. Then for each $i = 1, \dots, n$ the set $V(G_i)$ is m^3-convex in G.*

(2) *Let G be a HHP-free graph and $(v_1, \dots, v_n)$ be a MCS-ordering of G. Then for each $i = 1, \dots, n$ the set $V(G_i)$ is m^3-convex in G.*

Proof. We prove assertion (1) by induction on i. Assume that $V(G_i)$ is not m^3-convex in G but $V(G_j)$ is so for $j \geq i+1$. Then there must be a vertex y in G_{i+1} and an induced path P of length at least 3 connecting v_i and y, which contains some vertices not in G_i. Choose y and P such that $|P|$ is minimum and y is rightmost in the LexBFS-ordering.

Case 1. The neighbor of y in P does not belong to G_i.

Let x be this neighbor of y, and let $P = v_i - u_1 - \cdots - u_l - x - y$, $l \geq 1$. By applying $(P1)$ to $x < v_i < y$, we obtain a vertex $v > y$ adjacent to v_i but not to x.

The path $Q = v - v_i - u_1 - \cdots - u_l - x - y$ has both endpoints in G_{i+1}. By the induction hypothesis $V(G_{i+1})$ is m^3-convex. Thus Q cannot be induced. Since P is induced, all possible chords of Q must be incident to v. If v is adjacent only to y, we obtain a forbidden induced cycle of length at least 5. So let u_j be the vertex of $P \setminus \{y\}$ closest to y on the path P and adjacent to v. We immediately conclude $j = l$ for otherwise we have a hole. Now the m^3-convexity applied to $v - u_l - x - y$ implies $vy \in E$. Since the house and domino are forbidden subgraphs we conclude $l \geq 3$ (see Figure 2). Let $j < l$ be the index such that $vu_j \in E$, but $vu_s \notin E$ for all

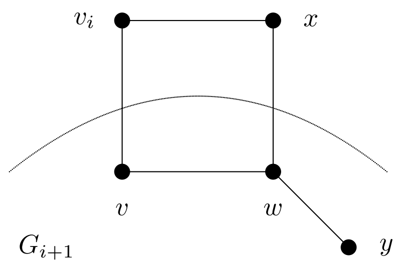

FIG. 3.

$s = j+1, \ldots, l-1$. For $j = l-1$ we have a house; for $j = l-2$ we obtain a domino; otherwise $v - u_j - \cdots - u_l - v$ forms a hole.

Case 2. The neighbor of y in P belongs to G_i.

By minimality of $|P|$ we immediately conclude $P = v_i - x - w - y$, where $w, y \in V(G_{i+1})$ and $x \notin V(G_i)$. Now $(P1)$ applied to $x < v_i < w$ gives a vertex $v > w$ adjacent to v_i but not to x. We may choose v with maximum number in the LexBFS-ordering. By considering the path $v - v_i - x - w$ the m^3-convexity implies $vw \in E$. Note that $vy \notin E$ for otherwise we obtain a house. Therefore, we have constructed a "P" (see Figure 3).

Case 2.1. $y < v$.

By applying $(P1)$ to $v_i < y < v$ we obtain a vertex $u > v$ adjacent to y but not to v_i. Note that $w < v < u$ implies $u \neq w$. Suppose $ux \in E$. Then $(P1)$ applied to $x < v_i < u$ gives a vertex $t > u > v$ adjacent to v_i but not to x, a contradiction to the maximality of v. Thus $ux \notin E$. In the path $v - w - y - u$ both endpoints have greater numbers than y. Let $y = v_j$ for some $j > i$. Then the m^3-convexity of G_{j+1} implies $uv \in E$ or $uw \in E$. If we have both edges, then we obtain a house induced by $\{v_i, x, v, w, u\}$. If $uv \in E$ but $uw \notin E$ then we have a domino. Finally, if $uw \in E$ and $uv \notin E$ then we can replace y by $u > y$ in P, a contradiction to the choice of y.

Case 2.2. $y > v$.

By applying $(P1)$ to $w < v < y$ we obtain a vertex $u > y$ adjacent to v but not to w. If $uv_i \in E$ then m^3-convexity implies the edges ux and uy. So $\{v_i, u, x, y, w\}$ induces a house. Thus $uv_i \notin E$. Moreover, with the same arguments as in Case 2.1 we show $ux \notin E$. In the path $u - v - w - y$ both endpoints have greater numbers than v. Let $v = v_j$ for some $j > i$. Then the m^3-convexity of G_{j+1} implies $uy \in E$. Thus we get a domino. This settles the proof of assertion (1).

Now to get a proof for assertion (2) we can repeat the arguments of the proof above up to Cases 2.1 and 2.2 using $(P2)$ instead of $(P1)$. $\square$

Note that any vertex $u \in V \setminus V(G_i)$ is semisimplicial in $G(\{u, v_i, \ldots, v_n\})$ since $V(G_i)$ is m^3-convex in G. Thus we can conclude the following.

COROLLARY 2.4 (see [19]).

(1) *For any HHD-free graph G and any LexBFS-ordering $(v_1, \ldots, v_n)$ of G vertex v_i is semisimplicial in G_i, $i = 1, \ldots, n$.*

(2) *For any HHP-free graph G and any MCS-ordering $(v_1, \ldots, v_n)$ of G vertex v_i is semisimplicial in G_i, $i = 1, \ldots, n$.*

Moreover, since there is a MCS-ordering of the "P," which is not a semisimplicial ordering and neither holes nor a domino contain a semisimplicial vertex we immediately conclude the following.

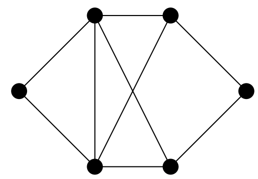

Fig. 4.

Theorem 2.5 (see [19]). *A graph G is HHP-free iff any MCS-ordering of any induced subgraph F of G is a semisimplicial ordering of F.*

Note that in Theorem 2.5 it is necessary to consider all induced subgraphs of a given graph, since the graph presented in Figure 4 contains a "P" but every MCS-ordering of this graph is a semisimplicial ordering. For LexBFS it is sufficient to consider the graph itself, since as we will show the class of graphs where any LexBFS-ordering gives a semisimplicial ordering is a hereditary class.

A graph is called *nontrivial* if it has at least two vertices.

Theorem 2.6. *The following conditions are equivalent for a graph G:*

(1) *G is HHD-free.*
(2) *Any LexBFS-ordering of G is a semisimplicial ordering.*
(3) *For any LexBFS-ordering $(v_1, \dots, v_n)$ of G the set $V(G_i)$ is m^3-convex in G for all $i = 1, \dots, n$.*
(4) *Every nontrivial induced subgraph of G has at least two semisimplicial vertices.*

Proof. It is easy to verify that none of a house, a domino, and holes contains two semisimplicial vertices. We have to show (2) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (4). All other directions are trivial or follow from Theorem 2.3.

(2) $\Longrightarrow$ (1) Let G be a graph such that every LexBFS-ordering is a semisimplicial ordering. Clearly, G cannot contain a hole or a domino since these graphs do not have a semisimplicial vertex. Assume that G contains a house induced by $\{a, b, c, d, e\}$ where $b - c - d - e - b$ induces a C_4 and a is adjacent to b and c. We start LexBFS at vertex a. By the rules of LexBFS both vertices d, e are smaller than the vertices b, c. Let v_i be the smaller one of d and e. Then v_i is not semisimplicial in G_i. Thus G is HHD-free.

(2) $\Longrightarrow$ (4) Let H be a nontrivial induced subgraph of G. Since H is HHD-free by (1) $\Longleftrightarrow$ (2) there must be some semisimplicial vertex v of H. Now starting procedure LexBFS at v gives a second semisimplicial vertex. ∎

Corollary 2.7. *Let G be a HHD-free graph and v be a vertex of G. Then there is a semisimplicial vertex u such that $d(u, v) = e(v)$.*

Proof. We start procedure LexBFS at v. The first vertex u of the obtained LexBFS-ordering is semisimplicial by the above theorem and fulfills $d(u, v) = e(v)$ by the rules of LexBFS. ∎

We immediately conclude the following.

Corollary 2.8. *In any nontrivial HHD-free graph G there is a pair of semisimplicial vertices u, v such that $d(u, v) = diam(G)$.*

Theorem 2.9. *The following conditions are equivalent for a graph G:*

(1) *G is HHD-free.*
(2) *The disk $D(v, 1)$ is m^3-convex for all vertices $v \in V$.*
(3) *The disks $D(v, k)$, $k \geq 1$, are m^3-convex for all vertices $v \in V$.*

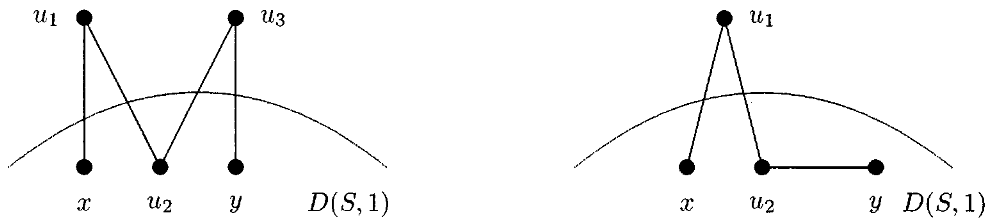

FIG. 5.

(4) *The set* $D(S,1) = \bigcup_{v \in S} D(v,1)$ *is* m^3*-convex for all connected sets* $S \subseteq V$.
(5) *The sets* $D(S,k) = \bigcup_{v \in S} D(v,k)$, $k \geq 1$, *are* m^3*-convex for all connected sets* $S \subseteq V$.

Proof. In every forbidden subgraph there is a vertex v such that $D(v,1)$ is not m^3-convex. So, we have to show only (1) $\Longrightarrow$ (5).

Suppose that there is a connected set S such that $D(S,1)$ is not m^3-convex. Then there are vertices x, y in $D(S,1)$ and there is an induced path $P = x - u_1 - \cdots - u_k - y$ such that $k \geq 2$ and at least one vertex u_i is not in $D(S,1)$. We may choose x, y, and P such that $|P|$ is minimal.

Case 1. $P \setminus \{x,y\} \subseteq V \setminus D(S,1)$.

We immediately conclude $x, y \notin S$. Moreover no u_i, $i = 1, \ldots, k$, is adjacent to some vertex of S. Let Q be a shortest path in $G(\{x,y\} \cup S)$ connecting x and y. Since $Q \setminus \{x,y\}$ is completely contained in S and both P and Q are induced, the cycle C formed by P and Q is chordless. From $|P| \geq 4$ we conclude $|C| \geq 5$—a contradiction.

Case 2. $|D(S,1) \cap P| \geq 3$.

By minimality of $|P|$, we obtain $k = 3$, $u_1, u_3 \notin D(S,1)$, and $u_2 \in D(S,1)$ or $k = 2$, $u_1 \notin D(S,1)$, and $u_2 \in D(S,1)$ (see Figure 5). Let $Q = x - z_1 - \cdots - z_l - y$, $l \geq 1$, be a shortest path in $G(\{x,y\} \cup S)$ connecting x and y and define $Q' := Q \setminus \{x,y\}$.

First consider the case $k = 2$. Note that $x, u_2 \notin S$, and u_1 is not adjacent to any vertex of Q'. Since the cycle $x - u_1 - u_2 - y - z_l - \cdots - z_1 - x$ is of length at least 5 the cycle lemma applied to the edge xu_1 gives $z_1 u_2 \in E$. If $yz_1 \in E$ then we have a house. Hence $l \geq 2$. If $u_2 z_2 \in E$ then we obtain a house. So let $u_2 z_2 \notin E$. If y is adjacent to z_2 then we have a domino. Thus $l \geq 3$ and we can apply the cycle lemma to the edge $z_1 u_2$ in the cycle $u_2 - y - z_l - \cdots - z_1 - u_2$ of length at least 5. So we conclude $u_2 z_3 \in E$ which gives a domino.

Now consider the case $k = 3$. Note that $x, y, u_2 \notin S$. Since Q' is completely contained in S neither u_1 nor u_3 is adjacent to any vertex of Q'. On the other hand, the cycle $x - u_1 - u_2 - u_3 - y - z_l - \cdots - z_1 - x$ is of length at least 6. Thus the cycle lemma applied to the edge $u_3 y$ implies $u_2 z_l \in E$. If $z_l x \notin E$ we proceed as in the case $k = 2$; otherwise we obtain a domino.

Thus, for every connected set S, $D(S,1)$ is m^3-convex. It is easy to see that $D(S,1)$ is connected too. Now, since $D(S,k) = D(D(S,k-1),1)$, we are done by induction on k. $\square$

COROLLARY 2.10. *If in a HHD-free graph nonadjacent vertices* $x, y \in N^k(v)$ *are joined by a path* P *such that* $P \setminus \{x,y\}$ *is contained in* $V \setminus D(v,k)$, *then there is a common neighbor of* x *and* y *in* $N^{k+1}(v) \cap P$.

3. Weak bipolarizable graphs. Here we characterize weak bipolarizable graphs as the graphs for which the m^3-convex alignment is a convex geometry. Let $\mathfrak{M}^3(G)$

denote the set of all m^3-convex sets of a graph G. For a set $S \subseteq V$ the *m^3-convex hull m^3-conv(S)* is the smallest member of $\mathfrak{M}^3(G)$ containing S.

A set $H \subseteq V$ is *homogeneous* iff $N(x) \setminus H = N(y) \setminus H$ for any pair of vertices x, y of H. A homogeneous set H is *proper* iff $1 < |H| < |V|$.

The next lemma gives a nice criterion for checking the semisimpliciality of a vertex.

LEMMA 3.1. *A vertex v of a graph G is semisimplicial in G iff the connected components of the complement of $G(N(v))$ are homogeneous in G.*

Proof. If v is not semisimplicial then there is a P_4 containing v as midpoint, say $u_1 - v - u_2 - u_3$. Now u_1 and u_2 belong to a common connected component C of the complement of $G(N(v))$. But C is not homogeneous in G due to u_3.

To prove the converse let C be a connected component of the complement of $G(N(v))$ and suppose that C is not homogeneous in G. Then there must be vertices $x, y \in C$ and a vertex $z \in V \setminus C$ such that $xz \in E$ but $yz \notin E$. We may choose x and y such that their distance in the complement of $G(C)$ is minimal. Obviously, $z \neq v$. Moreover, since $yz \notin E$ but every vertex from $N(v) \setminus C$ must be adjacent to every vertex of C, we have $z \notin N(v)$. Thus $z \in N^2(v)$. If $xy \notin E$ then $z - x - v - y$ is a P_4. If $xy \in E$ then let $x - u_1 - \cdots - u_k - y$ be a shortest path in the complement of $G(C)$. Thus $xu_1 \notin E$. The minimal distance of x, y now implies $u_1 z \notin E$. Therefore, $z - x - v - u_1$ is a P_4. ∎

THEOREM 3.2 ([23]). *A graph G is weak bipolarizable iff each induced subgraph F of G is chordal or contains a proper homogeneous set.*

Let H be a proper homogeneous set in G and $v \in H$. Then the *homogeneous reduction $HRed(G, H, v)$* is the graph induced by $V(G) \setminus (H \setminus \{v\})$. Conversely, the *homogeneous extension $HExt(G, v, H)$* of G via a graph H in v with $V(H) \cap V(G) = \varnothing$ is the graph obtained by substituting v by H such that the vertices of H have the same neighbors outside of H as v had in G.

LEMMA 3.3. *Let H be a proper homogeneous set of a HHD-free graph G and $v \in H$.*

(1) *If x is semisimplicial in $HRed(G, H, v)$, but not in G, then $x \in H$, i.e., $x = v$.*
(2) *If $x \in H$ is semisimplicial in H, but not in G, then no vertex of H is semisimplicial in G and v is not semisimplicial in $HRed(G, H, v)$.*

Proof. Since no P_4 contains a proper homogeneous set, we conclude that for any 4-path P of G, either $P \subseteq H$ or $|P \cap H| \leq 1$.

(1) Since x is not semisimplicial in G it must be a midpoint of some 4-path P. If $x \notin H$ then the semisimplicity of x in $HRed(G, H, v)$ implies $|P \cap H| = 1$. But now we can replace the vertex of $P \cap H$ by v obtaining a P_4 in $HRed(G, H, v)$, which contains x as a midpoint—a contradiction. Thus $x \in H$, i.e., $x = v$.
(2) If $x \in H$ is semisimplicial in H, but not in G, then no P_4 in G containing x as a midpoint is completely contained in H. Thus $P \cap H = \{x\}$ for any 4-path P in G with midpoint x. Since H is homogeneous we can replace x in P by any vertex of H. Thus no vertex of H is semisimplicial in G, and v is not semisimplicial in $HRed(G, H, v)$. ∎

In [16] it is proved that in a chordal graph every nonsimplicial vertex lies on an induced path between two simplicial vertices. Next we present a stronger result which we will subsequently use.

LEMMA 3.4. *Let G be a chordal graph and $P = v_1 - \cdots - v_k$ be an induced path of length at least 2, i.e., $k \geq 3$. Then there are vertices u_i, $i = 1, \ldots, s$ and w_j, $j = 1, \ldots, t$, such that u_1, w_1 are simplicial and $u_1 - u_2 - \cdots - u_s - v_2 - \cdots - v_{k-1} -$*

$w_t - \cdots - w_2 - w_1$ *is an induced path in* G.

Proof. If both v_1 and v_k are simplicial then we are done. So suppose that v_1 is not simplicial.

Let M be the m-convex hull of $\{v_1, \ldots, v_k\}$ and S be the neighborhood of v_1 in M. Obviously, S is a $v_1 - v_3$-separator in M, i.e., v_1 and v_3 are in different connected components of $G(M) \setminus S$. We show that S is a $v_1 - v_3$-separator in G too. Assuming the contrary there must be an induced path P in $V \setminus S$ joining v_1 and v_3. Since S is the set of neighbors of v_1 in M the neighbor of v_1 in P does not belong to M. Thus P is an induced path between vertices of M which contains vertices of $V \setminus M$, a contradiction to the m-convexity of M. Therefore, S is a $v_1 - v_3$-separator in G.

Recall that every chordal graph is either complete or contains at least two nonadjacent simplicial vertices [7, 24]. Thus $G(M)$ as a chordal graph must contain at least two simplicial vertices. Since deleting a simplicial vertex from a m-convex set preserves m-convexity and since M is the m-convex hull of $\{v_1, \ldots, v_k\}$ we immediately conclude that v_1 and v_k are the only two simplicial vertices of M. Thus S is complete.

Since v_1 is not simplicial and all neighbors of v_1 are contained in $F := G(K \cup S)$, where K is the connected component of $G \setminus S$ containing v_1, the chordal graph F is not complete and hence there are two nonadjacent simplicial vertices in F. By the completeness of S at most 1 of them is in S. Thus we have a simplicial vertex u_1 in K which is simplicial in G too. Now consider a path P connecting the vertices v_1 and u_1 in K. Then no vertex up to v_2 of an induced subpath $u_1 - \cdots - u_s - v_2$ of the path $P \cup v_1 v_2$ has a neighbor in $\{v_3, \cdots, v_k\}$. Hence, $u_1 - \cdots - u_s - v_2 - \cdots - v_k$ is an induced path. For v_k we proceed analogously. □

Note that every simplicial vertex is semisimplicial and thus, every nonsemisimplicial vertex is nonsimplicial.

LEMMA 3.5. *Every nonsemisimplicial vertex of a weak bipolarizable graph G lies on an induced path of length at least* 3 *between two semisimplicial vertices.*

Proof. We prove the assertion by induction on the size of G. The assertion holds for all graphs with at most 4 vertices since the only graph of these sizes which contains a nonsemisimplicial vertex is the P_4. Let x be a nonsemisimplicial vertex of G, i.e., x is a midpoint of some P_4.

If G is chordal then by Lemma 3.4 there is a path P of length at least 3 containing x such that both endpoints of P are simplicial and thus semisimplicial in G. Consequently, we are done.

Now assume that G is not chordal. Hence, by Theorem 3.2, G contains a proper homogeneous set H.

Case 1. $x \in H$.

Suppose that x is semisimplicial in $HRed(G, H, x)$. Then by Lemma 3.3 (2), vertex x is not semisimplicial in H. By the induction hypothesis x lies on an induced path of length at least 3 between semisimplicial vertices y, z in H. By Lemma 3.3 (2), both y and z must be semisimplicial in G too.

Now assume that x is not semisimplicial in $HRed(G, H, x)$. By the induction hypothesis x lies on an induced path between semisimplicial vertices y, z in $HRed(G, H, x)$. In particular, $y, z \notin H$. Thus by Lemma 3.3 (1), both y and z must be semisimplicial in G too.

Case 2. $x \notin H$.

From Lemma 3.3 (1) we immediately conclude that x is not semisimplicial in $HRed(G, H, v)$, where v is a semisimplicial vertex in the weak bipolarizable graph H.

By the induction hypothesis x lies on an induced path between semisimplicial vertices y, z in $HRed(G, H, v)$. Suppose that y is not semisimplicial in G. From Lemma 3.3 (1), we infer $y = v$. But now $y = v$ is not semisimplicial in $HRed(G, H, v)$ by Lemma 3.3 (2)—a contradiction. Thus both y and z are semisimplicial in G too. □

To prove the next corollary we use the arguments of the proof of [16, Corollary 3.4].

COROLLARY 3.6. *The Caratheodory number of the m^3-convex alignment of a weak bipolarizable graph is at most* 2.

Proof. Let $G = (V, E)$ be a weak bipolarizable graph and S be a subset of V. Pick an arbitrary vertex $x \in m^3$-$conv(S)$. If x is semisimplicial in the subgraph induced by m^3-$conv(S)$, then $x \in S$ since each extreme point of m^3-$conv(S)$ is in S by the definition of the hull of S. Otherwise, by Lemma 3.5, x lies on an induced path of length at least 3 between semisimplicial vertices of the subgraph induced by m^3-$conv(S)$. Hence, x is in the m^3-convex hull of two extreme points of m^3-$conv(S)$. Since each extreme point of m^3-$conv(S)$ is in S we are done. □

Subsequently, we call a vertex set S of G *reachable* iff there is an ordering $(v_1, \dots, v_k)$ of $V \setminus S$ such that for each $i = 1, \dots, k$ vertex v_i is semisimplicial in $G(\{v_i, \dots, v_k\} \cup S)$.

THEOREM 3.7. *The following conditions are equivalent for a graph G:*

(1) *G is weak bipolarizable.*
(2) *In every induced subgraph F of G each nonsemisimplicial vertex lies on an induced path of length at least* 3 *between semisimplicial vertices of F.*
(3) *Each m^3-convex set of G is the hull of its semisimplicial vertices, i.e., $(V(G), \mathfrak{M}^3(G))$ is a convex geometry.*
(4) *A set S of G is m^3-convex iff there is an ordering $(v_1, \dots, v_k)$ of $V(G) \setminus S$ such that for each $i = 1, \dots, k$ vertex v_i is semisimplicial in $G(\{v_i, \dots, v_k\} \cup S)$, i.e., S is reachable.*

Proof. We only need to prove (4) $\Longrightarrow$ (1).

Claim 1. If S is a m^3-convex set in $F := HRed(G, H, v)$, where H is a proper homogeneous set of G, then

$$S' := \begin{cases} S & : \quad v \notin S, \\ S \cup H & : \quad v \in S \end{cases}$$

is m^3-convex in G.

Suppose S' is not m^3-convex in G. Then there must be vertices $x, y \in S'$ and an induced path P of length at least 3 joining x and y such that $P \setminus S' \neq \varnothing$. If $|P \cap H| \leq 1$, then either P or $(P \setminus H) \cup \{v\}$ is an induced path in F of length at least 3 joining vertices of S which has at least one vertex outside S, a contradiction to the m^3-convexity of S in F. Now suppose $|H \cap P| \geq 2$. Note that $P \setminus H \neq \varnothing$. Let $P' = u_1 - \cdots - u_k$ be a maximal by inclusion subpath of P completely contained in H. Suppose $k \geq 2$. If $u_1 = x$ then $u_k \neq y$ since $P \setminus H \neq \varnothing$. Since H is homogeneous u_1 must be adjacent to the neighbor of u_k in $P \setminus P'$—a contradiction. If $u_1 \neq x$ then the same argument can be applied to u_k and the neighbor of u_1 in $P \setminus P'$. Now let $k = 1$. For $|H \cap P| \geq 2$ there must be a vertex $z \in H \cap P \setminus N(u_1)$. But now $N(u_1) \setminus H = N(z) \setminus H$ and $|P| \geq 4$ imply some chords in P, again a contradiction. Therefore, S' is m^3-convex in G.

Claim 2. Every homogeneous set H of a graph G is m^3-convex.

Let x, y be nonadjacent vertices of a homogeneous set H in G. If x has a neighbor z outside H then $yz \in E$, and vice versa. Thus any induced path between nonadjacent vertices of H containing vertices from $V \setminus H$ must be of length 2. Consequently, H is m^3-convex in G.

Claim 3. Let H be a proper homogeneous set of a graph G. If S is m^3-convex in $G(H)$ then it is so in G.

Since S is a subset of H we can use the same arguments as in the proof of Claim 2.

Claim 4. If v is a simplicial vertex in a graph G then any m^3-convex set of $G \setminus \{v\}$ is m^3-convex in G.

Since the neighborhood of a simplicial vertex v is complete no induced path of length at least 3 can contain v as an inner point.

Now we prove by induction on the size of G that any graph fulfilling (4) is weak bipolarizable, i.e., HHDA-free. Since any singleton of $V(G)$ is a m^3-convex set, G possesses a semisimplicial ordering, and thus does not contain a hole or a domino. Let F be an induced subgraph of G isomorphic to the house and K be the 3-clique of F. Now the m^3-convex set K must be reachable, but no vertex of $F \setminus K$ is semisimplicial in F—a contradiction. Therefore, G is a HHD-free graph.

Case 1. G contains a proper homogeneous set H.

Let v be a vertex of H, $F := HRed(G, H, v)$ and S be a m^3-convex set in F. Then S' as defined in Claim 1 is m^3-convex in G and thus reachable. Hence, S is reachable in F since each semisimplicial vertex of G is semisimplicial in every induced subgraph containing this vertex. Therefore, F fulfills (4) and, by the induction hypothesis, is HHDA-free. Applying the same arguments to a m^3-convex set S of H and using Claim 3 implies that H is HHDA-free. Now we conclude that G itself is HHDA-free as the homogeneous extension of the HHDA-free graph F by the HHDA-free graph H (see [23]).

Case 2. G has no proper homogeneous set.

Suppose G contains an "A" induced by the 4-cycle $x - c - d - y - x$ and the pendant vertices a, b where $ax \in E$ and $by \in E$. In what follows we prove that $M := D(a, 1) \cup D(b, 1)$ is m^3-convex in G. Thus M must be reachable, but neither c nor d are semisimplicial in the "A"—a contradiction.

First note that every semisimplicial vertex v of G is simplicial due to Lemma 3.1. From Claim 4 we conclude that $G \setminus \{v\}$ fulfills (4) and thus, by the induction hypothesis, is HHDA-free. Therefore, a and b are the only semisimplicial vertices of G, and $D(a, 1)$, $D(b, 1)$ are complete.

- *If there is a common neighbor z of a and b, then z is adjacent to all vertices a, b, c, d, x, y.*
  Considering the cycle $z - a - x - y - b - z$ implies the edges zx and zy. Now $\{z, x, y, c, d\}$ induces a house, thus $zc \in E$ or $zd \in E$. Suppose $zc \notin E$. Then $zd \in E$ and $\{a, z, x, c, d\}$ induces a house. Hence both $zc \in E$ and $zd \in E$.
- *$N(a) \subseteq N(c)$ and $N(b) \subseteq N(d)$.*
  Let w be a neighbor of a and suppose $wc \notin E$. Thus $w \neq x$, $wx \in E$, and $wb \notin E$. Since $G \setminus \{a\}$ is HHDA-free w must be adjacent to y or d. If $wy \in E$ then the graph induced by $\{w, x, y, c, d\}$ implies $wd \in E$. Hence $wd \in E$. But now $\{a, x, w, c, d\}$ induces a house.
- *Every vertex of $N(a)$ is adjacent to every vertex of $N(b)$.*
  If $w \in N(a) \cap N(b)$, then w is adjacent to all vertices of $N(a) \cup N(b)$ since both $D(a, 1)$ and $D(b, 1)$ are complete. So suppose for the contrary that there are nonadjacent vertices $z \in N(a) \setminus N(b)$ and $w \in N(b) \setminus N(a)$. Since $xy \in E$ we have either $z = x$ and $w \neq y$, $z \neq x$, and $w = y$ or $z \neq x$ and $w \neq y$.

First assume $z = x$ (analogously, $w = y$). The graph induced by $\{w, d, y, c, z\}$ implies $wc \in E$. But now $\{b, y, w, z, c\}$ induces a house. So let $x \neq z$ and $y \neq w$. By the same arguments as above we may assume $zy \in E$ and $wx \in E$. Now considering $\{w, d, y, z, c\}$ gives $zd \in E$ or $wc \in E$. By symmetry, say $wc \in E$. But this yields a house induced by $\{b, y, w, z, c\}$.

To complete the proof suppose that $M = D(a, 1) \cup D(b, 1)$ is not m^3-convex in G. Then there must be nonadjacent vertices $w, z \in M$ and an induced path P of length at least 3 joining w and z such that $P \setminus M$ is nonempty. Since every vertex of $N(a)$ is adjacent to every vertex of $N(b)$ we conclude $\{w, z\} \cap \{a, b\} \neq \varnothing$. Say $z = a$. Then $w \notin D(a, 1)$. Let z' be the neighbor of z in P, i.e., $z' \in N(a)$. If $w \in N(b)$ then $z'w \in E$ gives a contradiction. Hence $w = b$. Now consider the neighbor w' of w in P. From $w' \in N(b)$ we conclude $z'w' \in E$—again a contradiction. $\square$

4. The existence of r-dominating cliques. Let $r : V \to \mathbb{N}$ be some vertex function defined on G. Then a set $D \subseteq V$ *r-dominates* G iff for all vertices x in $V \setminus D$ there is a vertex $y \in D$ such that $d(x, y) \leq r(x)$. D is a *r-dominating clique* iff D is complete and r-dominates G. Note that there are graphs and vertex functions r such that G has no r-dominating clique. For some graph classes, such as chordal, distance-hereditary, and HHDS-free graphs, there is an existence criterion for r-dominating cliques [9, 8, 10]. In what follows we prove this criterion for HHD-free graphs. The method is similar to the one used for chordal graphs in [9] and essentially exploits m^3-convexity of disks in HHD-free graphs.

LEMMA 4.1. *Let C be a clique in a HHD-free graph G and v be a vertex of G such that for all vertices w of C the distance to v is $k \geq 1$. Then there is a vertex u at distance $k - 1$ to v which is adjacent to all vertices of C.*

Proof. We prove the assertion by induction on k. For $k = 1$ there is nothing to show. Let x be a vertex of $N^{k-1}(v)$ adjacent to a maximal number of vertices of C. Suppose that there is some vertex $a \in C$ which is not adjacent to x, and let y be a neighbor of a in $N^{k-1}(v)$. By the choice of x there must be a vertex $b \in C$ adjacent to x but not to y. Thus we have the path $x - b - a - y$ of length 3 between vertices x, y of $D(v, k-1)$, which contains vertices a, b outside of $D(v, k-1)$. By Theorem 2.9 $D(v, k - 1)$ is m^3-convex; hence $xy \in E$. Now, by applying the induction hypothesis to the clique $\{x, y\}$ we obtain a common neighbor u of x, y in $N^{k-2}(v)$. Therefore we have constructed a house—a contradiction. $\square$

In a similar way we can prove the following lemma.

LEMMA 4.2. *If x, y, v are vertices of a HHD-free graph G such that $d(x, v) = d(y, v) = k$ and $N(x) \cap N(y) \cap N^{k+1}(v) \neq \varnothing$, then there is a vertex $u \in N(x) \cap N(y) \cap N^{k-1}(v)$.*

Define the *projection* of a vertex v to a set S by

$$Proj(v, S) := \{x \in S : d(v, x) = d(v, S)\}$$

and the projection of a set C to a set S by $Proj(C, S) := \bigcup_{v \in C} Proj(v, S)$.

LEMMA 4.3. *Let u, v be vertices of a HHD-free graph. Then for any vertex x in $D(v, k)$ there is a shortest path between u and x going through the projection $Proj(u, D(v, k))$.*

Proof. If $d(u, v) \leq k$ then $Proj(u, D(v, k)) = \{u\}$ and there is nothing to show. So let $d(u, v) \geq k + 1$. Choose an arbitrary vertex $w \in Proj(u, D(v, k))$ and assume $d(u, x) < d(u, w) + d(w, x)$. Let P be a shortest path connecting u and x, and let z be the vertex of $V(P) \cap D(v, k)$ closest to u on the path P (see Figure 6). Thus

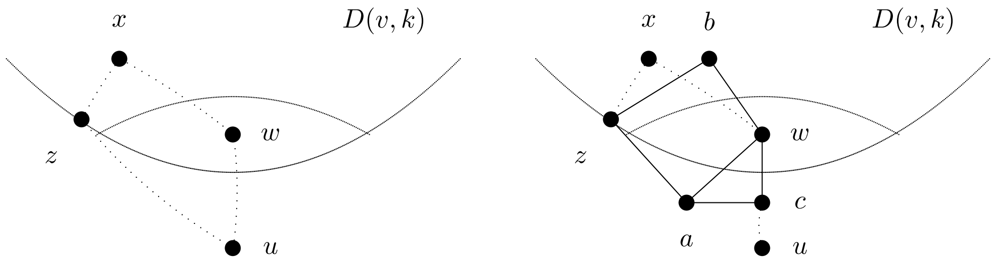

FIG. 6.

$d(u,x) = d(u,z) + d(z,x)$. If $z \in Proj(u, D(v,k))$ then we are done. So assume $z \notin Proj(u, D(v,k))$ implying $d(u,z) > d(u,w)$. Note that $zw \notin E$ for, otherwise,

$$d(u,w)+1+d(z,x) \leq d(u,z)+d(z,x) = d(u,x) < d(u,w)+d(w,x) \leq d(u,w)+d(z,x)+1$$

is a contradiction. Thus by Corollary 2.10 there is a common neighbor a of w and z in $N^{k+1}(v) \cap P$ implying that $d(u,z) \leq d(u,w) + 2$ and $d(w,x) \leq d(z,x) + 2$. Moreover, $d(u,x) < d(u,w) + 2 + d(z,x)$. Therefore, $d(u,z) + d(z,x) = d(u,x) < d(u,w) + 2 + d(z,x)$ gives $d(u,z) = d(u,w) + 1$, and $d(u,a) = d(u,w)$. Now applying Lemma 4.2 to z, w, and v gives a common neighbor b of z, w in $N^{k-1}(v)$. By distance requirements $ab \notin E$. Furthermore, Lemma 4.1, applied to $\{a, w\}$ and u, yields a common neighbor c of a, w at distance $d(u,w) - 1$ to u. Thus neither $cz \in E$ nor $cb \in E$. Consequently, $\{a, b, c, w, z\}$ induces a house. □

Let U_1, U_2 be subsets of V. The sets U_1, U_2 form a *join* iff any vertex of U_1 is adjacent to any vertex of U_2.

LEMMA 4.4. *Let G be a HHD-free graph and xy be an edge outside of $D(v,k)$. Then $Proj(x, D(v,k)) \subseteq Proj(y, D(v,k))$ or $Proj(y, D(v,k)) \subseteq Proj(x, D(v,k))$. Moreover, assuming $Proj(x, D(v,k)) \subseteq Proj(y, D(v,k))$ implies that the sets $Proj(x, D(v,k))$ and $Proj(y, D(v,k)) \setminus Proj(x, D(v,k))$ form a join.*

Proof. We will present the proof for the equidistant case, i.e., $d(x,v) = d(y,v)$. The cases $d(x,v) = d(y,v)+1$ and $d(y,v) = d(x,v)+1$ can be handled in a similar (even easier) way. Let $A := Proj(x, D(v,k)) \cap Proj(y, D(v,k))$, $B := Proj(x, D(v,k)) \setminus A$, and $C := Proj(y, D(v,k)) \setminus A$.

Suppose $w_x \in B$, $w_y \in C$. Since $d(y, w_y) = d(x, w_x) = d(x,v) - k$ we have $d(x, w_y) = d(x, w_x) + 1$ and $d(y, w_x) = d(y, w_y) + 1$. Now if $w_x w_y \notin E$ we get a contradiction to Corollary 2.10. Therefore, $w_x w_y \in E$. Let b (c) be the neighbor of w_x (w_y) in a shortest path P_x (P_y) between x (y) and w_x (w_y). Obviously, $w_x c, w_y b \notin E$. Lemma 2.2 applied to the edge $w_x w_y$ in the cycle induced by the vertices of P_x and P_y gives $bc \in E$. Thus $\{b, c, w_x, w_y, s\}$ induces a house where s is a common neighbor of $w_x w_y$ in $N^{k-1}(v)$ due to Lemma 4.1. Consequently, either $B = \varnothing$ or $C = \varnothing$.

Finally, suppose $w \in A$, $w_x \in B$ and $w_x w \notin E$. Consider the three vertices w, w_x, v. By Corollary 2.10 there is a common neighbor z of w and w_x at distance $k+1$ to v and $d(x,w) - 1$ to x. By Lemma 4.2 there is a common neighbor u of w and w_x at distance $k-1$ to v. Let t be the neighbor of w on a shortest path joining w and y. Since $w_x \notin A$ we have $tw_x \notin E$. By distance requirements $zu, tu \notin E$. If $tz \in E$ then $\{t, z, w, w_x, u\}$ induces a house. So assume $tz \notin E$ and consider the cycle C formed by w and by the shortest paths joining t, y and z, x. Obviously $|C| \geq 5$. Applying the circle lemma to edge zw yields the edge ts, where s is the neighbor of

z in the shortest path between x and z. By distance requirements $\{s, t, z, w, w_x, u\}$ induces a domino. Therefore, A and B form a join. □

LEMMA 4.5. *Let G be a HHD-free graph and C be a clique such that $C \setminus D(v,k) \neq \varnothing$. Then there is some vertex $u \in N^{k-1}(v)$ adjacent to all vertices of $Proj(C, D(v,k))$.*

Proof. Choose a maximal clique C' in $Proj(C, D(v,k))$ containing $C \cap D(v,k)$. By Lemma 4.1 there is a vertex a in $N^{k-1}(v)$ adjacent to all vertices of C'. Choose such a vertex a with a maximal number of neighbors in $Proj(C, D(v,k))$ and suppose that there is some vertex $y \in Proj(C, D(v,k)) \setminus C'$ nonadjacent to a. Since C' is maximal there is a vertex $w \in C'$ which is not adjacent to y. Note $y \notin C$. Thus there is a common neighbor z of y and w in $N^{k+1}(v)$ (either $z \in C$ or the existence of z follows from Corollary 2.10). Now applying Lemma 4.2 to w, y gives a common neighbor b of w and y in $N^{k-1}(v)$. By distance requirements $za, zb \notin E$. If $ab \in E$, then $\{a, b, y, z, w\}$ induces a house. If $ab \notin E$, then we can apply Lemma 4.2 to a, b yielding a common neighbor c of a, b in $N^{k-2}(v)$. But now $\{c, a, b, y, w, z\}$ induces a domino. □

THEOREM 4.6. *Let G be a HHD-free graph and $r : V \to \mathbb{N}$ be a vertex function on G. Then G has a r-dominating clique iff for all vertices $x, y \in V$, $d(x,y) \leq r(x) + r(y) + 1$ holds.*

Proof. Obviously, if G has a r-dominating clique then the inequality is fulfilled. To prove the converse let $(v_1, \ldots, v_n)$ be any ordering of V and suppose that there is a clique C which r-dominates $\{v_1, \ldots, v_{i-1}\}$ but not v_i. Thus $d(v_i, C) \geq r(v_i) + 1$. Let $B := Proj(C, D(v_i, r(v_i) + 1))$.

Claim 1. B r-dominates $\{v_1, \ldots, v_{i-1}\}$.

Let $k \leq i-1$ and consider vertex v_k. Since C r-dominates $\{v_1, \ldots, v_{i-1}\}$ there is some vertex $c \in C$ such that $d(c, v_k) \leq r(v_k)$.

If $v_k \in D(v_i, r(v_i) + 1)$ then by Lemma 4.3 there is a shortest path joining c and v_k going through B. Thus v_k is r-dominated by some vertex of B.

Now let $v_k \in V \setminus D(v_i, r(v_i)+1)$. Since $d(v_k, v_i) \leq r(v_k) + r(v_i) + 1$ we may choose a vertex x_k in $D(v_k, r(v_k)) \cap N^{r(v_i)+1}(v_i)$. Again, by Lemma 4.3 there is a shortest path joining c and x_k which contains a vertex of B, say y_k. If $d(c, x_k) \geq 3$, then $y_k \in D(v_k, r(v_k))$ since both c and x_k are contained in the m^3-convex set $D(v_k, r(v_k))$. If $cx_k \in E$ then either $c = y_k$ or $x_k = y_k$ and we are done since v_k is r-dominated by y_k. So let $d(c, x_k) = 2$. Again, if $c = y_k$ or $x_k = y_k$ then we are done. Thus let $c - y_k - x_k$ induce a P_3 and assume $d(v_k, y_k) > r(v_k)$. We immediately conclude $d(v_k, c) = d(v_k, x_k) = r(v_k)$ and $d(v_k, y_k) = r(v_k) + 1$. Thus Lemma 4.2 applied to c, x_k, and v_k gives a common neighbor a of c and x_k at distance $r(v_k) - 1$ to v_k. Since

$$d(v_i, y_k) = d(v_i, x_k) = d(v_i, a) - 1 = d(v_i, c) - 1 = r(v_i) + 1$$

applying Lemma 4.1 to the edge $x_k y_k$ and to v_i yields a common neighbor b of x_k and y_k at distance $r(v_i)$ to v_i. By distance requirements the set $\{a, b, x_k, y_k, c\}$ induces a house—a contradiction. Thus y_k r-dominates v_k and we are done.

Let C'' be a maximal clique in $Proj(C, D(v_i, r(v_i) + 1))$ such that $C'' \supset C \cap D(v_i, r(v_i) + 1)$. By Lemma 4.5 there is a vertex a in $N^{r(v_i)}(v_i)$ adjacent to all vertices of B. Define $C' := C'' \cup \{a\}$.

Claim 2. C' r-dominates $\{v_1, \ldots, v_i\}$.

Obviously, a r-dominates v_i. Suppose there is some vertex v_k, $k \leq i-1$ which is not r-dominated by C'. By Claim 1 v_k is r-dominated by B. More exactly, there is a vertex $c \in C$ and a vertex $y_k \in Proj(c, D(v_i, r(v_i) + 1)) \subseteq B \setminus C'$ both r-dominating

v_k. Since C'' is maximal there must be a vertex $w \in C''$ nonadjacent to y_k. By Lemma 4.4 both vertices y_k, w are contained in the projection of c.

Let z be a common neighbor of w and y_k at distance $d(c, w) - 1$ to c obtained from Lemma 4.2. If $d(y_k, c) \geq 3$ then the m^3-convexity of $D(v_k, r(v_k))$ implies $z \in D(v_k, r(v_k))$. We conclude $d(v_k, z) = d(v_k, y_k) = r(v_k)$. Now we can apply Lemma 4.1 to the edge $y_k z$ obtaining a common neighbor s of y_k and z at distance $r(v_k) - 1$ to v_k. By distance requirements $sw, sa, az \notin E$. Thus $\{s, w, a, z, y_k\}$ induces a house. In a similar way we can handle the case $c = z$. So assume $d(y_k, c) = 2$. If $z \in D(v_k, r(v_k))$ then we proceed as above. So by assuming $d(z, v_k) > r(v_k)$ we have $d(v_k, c) = d(v_k, y_k) = r(v_k)$ and $d(v_k, z) = r(v_k)+1$. Now we can apply Lemma 4.2 to c, y_k obtaining a common neighbor b of c, y_k at distance $r(v_k) - 1$ to v_k. By distance requirements $bw, ba \notin E$. Thus $\{c, b, z, y_k, a, w\}$ induces a domino.

Consequently we have constructed a clique which r-dominates $\{v_1, \ldots, v_i\}$. Induction on i settles the proof. □

COROLLARY 4.7. *For a HHD-free graph G we have $2rad(G) \geq diam(G) \geq 2(rad(G) - 1)$.*

Proof. Suppose that $diam(G) < 2(rad(G) - 1)$. Then by Theorem 4.6 for $r(v) := rad(G) - 2$, $v \in V$, there exists a r-dominating clique C in G. Hence, any vertex v of C has $e(v) \leq rad(G) - 1$, a contradiction to the definition of the radius. □

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