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DOMINATION IN QUADRANGLE-FREE HELLY GRAPHS*

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INTRODUCTION

Let $G = (V, E)$ be an ordinary connected graph endowed with the standard metric $d(x, y)$, which is equal to the number of edges in the shortest chain joining the vertices x, y. On the vertex set V define a nonnegative integer-valued function r: $V \rightarrow \mathbf{N} \cup \{0\}$. We say that the vertex x is dominated by the vertex y if $d(x, y) \leq r(x)$. The set $M \subseteq V$ is called r-dominating if every vertex $x \in V$ is dominated by some vertex from M. The problem of finding the least r-dominating set of the graph G is called the r-domination problem.

The r-domination problem is a natural model for many location problems, which require locating the least number of service facilities at the vertices of a graph so that each vertex (consumption site) is within a distance $r(x)$ from some service facility.

Various versions of this problem are rapidly developing in the recent period, especially the problem with $r(x) = 1$ for all $x \in V$, i.e., the ordinary problem of finding the dominating (externally stable) set of a graph. For arbitrary graphs, this problem is NP-complete. Moreover, the dominating set problem remains NP-complete even for regular planar graphs of degree 4 [1], bipartite planar graphs of maximum degree 3 [2], bipartite chordal graphs [3], triangulated graphs and tree chain intersection graphs [4], split Helly graphs [5], and 2-CUB graphs [6]. The complexity of the dominating set problem on so-called monogenic classes of graphs is considered in [2].

Effective solution of the dominating set problem for arbitrary graphs thus involves considerable difficulties, and the main focus has accordingly shifted to identification of narrow, yet sufficiently meaningful classes of graphs on which this problem is solvable in polynomial time. In particular, polynomial-time algorithms have been constructed for trees [7], degrees of trees [8], intersection graphs of directed chains in a tree [4], sequential-parallel graphs [9], k-trees [6], permutation graphs [10], cacti [11], 1-CUB graphs [6], circular-arc intersection graphs [12], cographs [2, 13], and strongly chordal graphs [14], including interval graphs.

A graph is called chordal (or triangulated) if every cycle of length greater than 3 has a chord, i.e., an edge joining two nonconsecutive vertices of the cycle. A k-sun S_k is a graph whose vertex set can be partitioned into two subsets $X = \{x_1,\dots,x_k\}$ and $Y = \{y_1,\dots,y_k\}$ so that the set X generates a complete subgraph and the set Y an empty subgraph, and for every x_i, y_j the vertex x_i is adjacent to y_j if and only if $i = j$ or $i = j + 1 \pmod{k}$. Chordal graphs that do not contain k-suns ($k \geq 3$) as induced subgraphs are called strongly chordal graphs [15].

For the k-domination problem ($r(x) = k$ for all $x \in V$, k is a natural number) polynomial-time algorithms are available on strongly chordal graphs [16] and chordal graphs without induced subgraphs of the form S_3 and $\overline{S}_3$ (the complement of S_3) [17].

The graph G^k whose vertex set is identical with the vertex set of the graph G and the vertices x, y are adjacent if and only if $1 \leq d(x, y) \leq k$ in the graph G is called the k-th degree of the graph G. The k-domination problem on strongly chordal graphs and chordal graphs without induced subgraphs S_3 and $\overline{S}_3$ has been reduced in [16, 17] to the covering problem of the graph G^{2k} by a minimum number of cliques. It has been shown that for a strongly chordal graph G the graph G^{2k} is triangulated and for an S_3- and $\overline{S}_3$-free chordal graph G^{2k} is weakly triangulated. A graph is called weakly triangulated if it is without induced cycles of length ≥ 5 and their complements [18]. The clique covering problem is effectively solvable on these graphs [18, 19].

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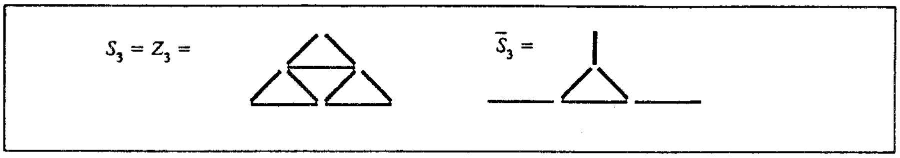

Fig. 1

A more general reduction of the r-domination problem to the clique covering problem for a special graph has been developed in [20]. We will need some definitions and notation for further discussion.

Thus, $B(x, r) = \{z \in V: d(x, z) \leq r\}$ denotes a ball of radius r ($r \in \mathbf{N} \cup \{0\}$) of the graph G centered at the vertex x, and $\mathfrak{N} = \{B(x, r): x \in V, r \in \mathbf{N} \cup \{0\}\}$, $\mathfrak{N}_1 = \{B(x, 1): x \in V\}$ is the family of all balls and the family of unit balls of the graph G.

The graph G is called a Helly graph [27] if the family of its balls $\mathfrak{N}$ has the Helly property, i.e., every subfamily of pairwise intersecting balls from $\mathfrak{N}$ has a nonempty intersection. The intersection graph $L(\mathfrak{M})$ of the family of sets $\mathfrak{M} = \{M_1,\dots,M_k\}$ is defined as follows: the vertices of the graph $L(\mathfrak{M})$ are the elements of $\mathfrak{M}$, and the vertices corresponding to the sets M_i, M_j are adjacent if and only if $M_i \cap M_j \neq \varnothing$. Clearly, $L(\mathfrak{N}_1) = G^2$.

There is a one-to-one correspondence between r-dominating sets of the Helly graph G and the clique coverings of the generated subgraphs of the graph $L(\mathfrak{N})$ constructed from G [20]. Thus, for effective solution of the r-domination problem on Helly graphs it suffices to be able to solve effectively the problem of least covering of the graph $L(\mathfrak{N})$ by complete subgraphs. In general Helly graphs (even split Helly graphs) the dominating set problem is NP-complete, as we have mentioned above. But if the graph $L(\mathfrak{N})$ is special, e.g., perfect, then the r-domination problem is solvable in polynomial time [21].

Triangulated Helly graphs in which the graphs $L(\mathfrak{N})$ are triangulated, weakly triangulated, or free from odd induced cycles of length ≥ 5 and their complements have been completely described in [22, 23] in terms of prohibited structures. As a particular case, we have established polynomial-time solvability of r-domination problems on strongly chordal graphs and graphs without generated subgraphs S_3 and $\bar{S}_3$. Note that a polynomial-time algorithm for the r-domination problem on strongly chordal graphs has been independently derived in [24].

All Helly graphs with triangulated $L(\mathfrak{N})$ are characterized in [5]. Here these graphs are called HT-graphs. HT-graphs are characterized in terms of dismantling by extremal vertices, whence it follows that strongly chordal graphs are precisely those graphs in which every generated subgraph is an HT-graph.

The subgraph H of the graph G is called isometric if the distance between any two vertices in H is equal to the distance between these vertices in G. A natural generalization of chordal graphs, i.e., graphs without induced cycles of length ≥ 4, are bridged graphs, i.e., graphs without isometric cycles of length ≥ 4 [25].

In this paper, the r-domination problem is considered on Helly graphs without quadrangles, i.e., without induced cycles of length 4. Alongside other results, we demonstrate polynomial-time solvability of the r-domination problem on bridged graphs without isometric k-suns S_k ($k \geq 3$) and bridged graphs without induced subgraphs S_3, $\bar{S}_3$. Recall [5] that even the dominating set problem is NP-complete on all quadrangle-free Helly graphs (Fig. 1).

CHARACTERIZATION

The graph G is called pseudomodular if for any vertices x, y, z of the graph G that satisfy the conditions $1 \leq d(x, y) \leq 2$, $d(x, z) = d(y, z) = k \geq 2$ there exists a vertex $v \in V$ such that $d(x, v) = d(y, v) = 1$, $d(z, v) = k - 1$ [26].

LEMMA 1 [27, 28]. G is a Helly graph if and only if G is a pseudomodular graph and the family $\mathfrak{N}_1$ of the graph G has the Helly property.

The subgraph Z_k ($k \geq 3$) of the graph G is called a flower of size k if its vertex set can be partitioned into two subsets $X = \{x_1,\dots,x_k\}$ and $Y = \{y_1,\dots,y_k\}$ so that the set X generates a complete subgraph in G and the set Y generates an empty subgraph, and for every x_i, y_j the vertex x_i is adjacent to y_j if and only if $i \neq j$.

A subgraph of the graph G is called suspended in G if there exists a vertex $v \in V$ which is adjacent to every vertex of this subgraph.

For the triangle (cycle of length 3) $\Delta = \{x, y, z\}$ of the graph G we denote by $\Gamma(\Delta)$ the set of vertices from V that are adjacent to at least two vertices of the triangle.

LEMMA 2 [28]. For a quadrangle-free graph G the following conditions are equivalent:

1) for any triangle Δ of the graph G there exists a vertex $v \in V$ such that $B(v, 1) \supset \Gamma(\Delta)$;

2) every flower in G is suspended.

The set $M \subseteq V$ of vertices of the graph G is called d-convex if together with any two vertices $x, y \in M$ it contains the d-segment $\langle x, y\rangle = \{z \in V: d(x, z) + d(z, y) = d(x, y)\}$ with endpoints at these vertices.

LEMMA 3 [20]. The balls of the graph g are d-convex if and only if for any vertices x, y, z of the graph G such that $d(x, y) \leq 3$, $z \in \langle x, y\rangle$, $d(x, v) = d(y, v)$, we have $d(v, z) \leq d(v, x)$.

In what follows, C_k and $\overline{C}_k$ $(k \geq 4)$ are an induced cycle of length k and its complement.

C_4-free Helly graphs can be characterized by the following theorem.

THEOREM 1. For the graph G conditions 1)-3) are equivalent:

1) G is a C_4-free Helly graph;

2) for any vertices x, y, z, v of the graph G such that $z \in \langle x, v\rangle$, $v \in \langle z, y\rangle$, $d(z, v) = 1$, we have $d(x, y) \geq d(x, z) + d(v, y)$, and every subgraph of the form C_5 and Z_k $(k \geq 3)$ is suspended in G;

3) every ball of the graph G is d-convex, and every subgraph of the form C_5 and Z_k $(k \geq 3)$ is suspended in G.

Proof. 1) $\Rightarrow$ 2). Since unit balls centered at the vertices of the cycle C_5 are pairwise intersecting, the total intersection of these balls is nonempty. We similarly prove that subgraphs of the form Z_k $(k \geq 3)$ are suspended.

Now let the vertices x, y, z, v be such that $z \in \langle x, v\rangle$, $v \in \langle z, y\rangle$, $d(z, v) = 1$, but $d(x, y) \leq d(x, z) + d(v, y) - 1$. In the graph G consider four pairwise intersecting balls $B(z, 2)$, $B(v, 1)$, $B(x, d(x, z))$, $B(y, d(v, y) - 1)$. By the Helly property, there exists a vertex $w \in \langle v, y\rangle$ adjacent to v such that $d(z, w) = 2$ and $d(z, x) = d(w, x)$. Again by pairwise intersection of the balls $B(z, 1)$, $B(w, 1)$, and $B(x, d(z, x) - 1)$ we obtain the existence of a vertex $u \in \langle z, w\rangle$ such that $d(u, x) = d(z, x) - 1$. But then the vertices z, v, w, u form a cycle C_4 in G. The contradiction completes the proof of the implication.

2) $\Rightarrow$ 3). It suffices to prove that the graph G satisfies the condition of Lemma 3. Let $d(x, y) \leq 3$, $v \in \langle x, y\rangle$, and $d(z, x) = d(z, y) < d(z, v)$ for the vertices $x, y, z, v \in V$. Without loss of generality we may assume that v, x are adjacent vertices. Then by the condition $x \in \langle z, v\rangle$ we should have the inequality $d(z, y) \geq d(y, v) + d(z, x)$. But this is impossible because $d(z, y) = d(x, z)$ and $1 \leq d(y, v) \leq 2$. The relationship $d(z, v) \leq d(z, x)$ thus holds for all vertices $v \in \langle x, y\rangle$.

3) $\Rightarrow$ 1). We first prove that the graph G is pseudomodular.

a) Let $x, y, z \in V$, $d(x, y) = 1$, $d(z, y) = d(z, x)$. Consider the vertices v, w that are respectively adjacent to x, y and are at a distance $d(z, x) - 1$ from z. By d-convexity of the ball $B(z, d(z, x) - 1)$ we have $d(v, w) \leq 2$ and $d(z, t) \leq d(z, v)$ for all $t \in \langle v, w\rangle$. Now assume that there does not exist a vertex u adjacent to x, y and such that $d(u, z) = d(z, x) - 1$. Since a graph with d-convex balls is C_4-free, we conclude that $d(v, w) = 2$ and the vertices x, y, v, w together with $t \in \langle v, w\rangle \setminus \{v, w\}$ form a cycle C_5. The vertex on which this cycle is suspended is adjacent to x, y and is contained in the d-segment $\langle v, w\rangle$. It is therefore at a distance $d(z, x) - 1$ from the vertex z — a contradiction. Thus there exists a vertex u adjacent to x, y for which $d(z, u) = d(z, x) - 1$.

b) Now let $x, y, z \in V$, $d(x, y) = 2$, $d(z, x) = d(z, y)$. Assume again that there does not exist a vertex $u \in \langle x, y\rangle \setminus \{x, y\}$ for which $d(u, z) = d(x, z) - 1$. Then by d-convexity of the ball $B(z, d(x, z))$ the vertex $v \in \langle x, y\rangle \setminus \{x, y\}$ satisfies $d(v, z) = d(z, x) = d(z, y)$. By the above there exist vertices u_1, u_2 such that u_1 is adjacent to x, v, u_2 is adjacent to y, v, and $d(z, u_1) = d(z, u_2) = d(z, x) - 1$. These vertices are adjacent to one another, as otherwise we would have a contradiction with d-convexity of the ball $B(z, d(x, z) - 1)$. Again there exists a vertex w adjacent to u_1, u_2 and located at a distance $d(z, x) - 2$ from z. It is easy to see that the vertices x, v, y, u_1, u_2, w generate in G a subgraph of the form Z_3, which by assumption is not suspended — a contradiction.

Thus, the graph G is pseudomodular. To complete the proof, we show that the family $\mathfrak{N}_1$ of the graph G has the Helly property. Let $\mathfrak{M} = \{B(y_1, 1), B(y_2, 1), \ldots, B(y_m, 1)\}$ be a family of $m \geq 3$ balls every $m - 1$ of which intersect and $x_i \in \{B(y_j, 1): j = 1,2,\ldots,m, i \neq j\}$.

Consider the triple of vertices x_1, x_2, x_3. Without loss of generality, we may assume that these are distinct vertices. If the vertices x_1, x_2, x_3 are pairwise not adjacent, then they are at a distance 2 from one another. By pseudomodularity of the graph, there exists a vertex v adjacent to x_1, x_2, x_3 simultaneously. Since the balls of the graph are d-convex, the vertex v is contained in all balls of the family $\mathfrak{M}$, and thus the total intersection of these balls is nonempty.

Now let x_1, x_2 be adjacent vertices. If the vertex x_3 is not equally distant from the vertices x_1, x_2, then it is adjacent to one of them, and this vertex, being adjacent to the other two vertices, is contained in all the balls simultaneously (by d-convexity of the balls). If $d(x_1, x_3) = d(x_2, x_3) = 2$, then by pseudomodularity of the graph there exists a vertex v which is adjacent to x_1, x_2, x_3 and is therefore contained in all the balls that contain the vertex x_3. Hence, it suffices to consider the case when the vertices x_1, x_2, x_3 are pairwise adjacent, i.e., form the triangle Δ.

Since the balls are d-convex,

$$x_1, x_2 \in \cap \{B(y_i, 1): i = 3, 4, \ldots, m\},$$

$$x_3, x_2 \in \cap \{B(y_i, 1): i = 1, 4, \ldots, m\},$$

$$x_1, x_3 \in \cap \{B(y_i, 1): i = 2, 4, \ldots, m\},$$

for the triangle $\Delta = \{x_1, x_2, x_3\}$ we have $\Gamma(\Delta) \supseteq \{y_1, y_2, \ldots, y_m\}$, and since every subgraph of the form Z_k $(k \geq 3)$ is suspended in G and G is C_4-free, by Lemma 2 there exists a vertex $v \in V$ such that $B(v, 1) \supseteq \Gamma(\Delta) \supseteq \{y_1, y_2, \ldots, y_m\}$. Thus in this case also the total intersection of the balls in $\mathfrak{M}$ is nonempty. Q.E.D.

COROLLARY 1. A bridged graph without isometric subgraphs S_3 is a Helly graph.

To prove the corollary, it suffices to note that bridged graphs are C_5-free, their balls are d-convex [25], and every flower Z_k $(k \geq 3)$ contains an isometric subgraph S_3.

3. SUBCLASSES OF HT-GRAPHS

A graph whose family of balls $\mathfrak{N}$ has the Helly property and whose graph $L(\mathfrak{N})$ is triangulated is called an HT-graph [5].

LEMMA 4. Let G be a Helly graph without isometric subgraphs of the form C_4 and S_k $(k \geq 4)$. Then the graph G^2 is triangulated.

Proof. The proof is by induction on the number of vertices. Since G is a Helly graph, there exist a pair of vertices $x, y \in V$ such that $B(x, 1) \subseteq B(y, 1)$ (see [27, 28]). It is easy to see that G is also a Helly graph without isometric subgraphs of the form C_4 and S_k $(k \geq 4)$. By the induction hypothesis, the graph $(G - x)^2$ is C_k-free $(k \geq 4)$. Assume that the graph G^2 contains a cycle C_k $(k \geq 4)$ generated by the vertices $x_1, x_2, \ldots, x_k$. Among all these cycles take the one with the least k. Without loss of generality, assume that $x_1 = x$. In the graph $(G - x)^2$ the vertices $y, x_2, \ldots, x_k$ form a simple cycle of length k. Since $(G - x)^2$ is free from subgraphs of the form C_k $(k \geq 4)$, this cycle is divided by chords into triangles. Note that the vertex y is one of the endpoints of all these chords.

We thus have $d(x_i, y) = d(x_i, x) - 1 = 2$ for all $i = 3, \ldots, k-1$; $d(x_i, x_j) \leq 2$ if and only if $j = i \pm 1 \pmod{k}$; $d(x_2, y) \leq 2$, $d(x_k, y) \leq 2$.

By the Helly property, for every triple y, x_i, x_{i+1}, $i = 2, 3, \ldots, k-1$, there exists a vertex v_i which is adjacent only with y, x_i, x_{i+1}. Since the graph G is C_4-free, each cycle $(y, v_i, x_{i+1}, v_{i+1}, y)$ contains the chord (v_i, v_{i+1}). Similarly for the 4-tuple x, y, x_2, v_2 there exists a vertex $v_1 \in B(x, 1) \cap B(y, 1) \cap B(x_2, 1) \cap B(v_2, 1)$. Without loss of generality, we assume that v_1 is neither x_2 nor y. Among all such combinations $M = \{v_1, v_2, \ldots, v_{k-1}\}$ choose the one in which the subgraph generated by these vertices has the maximum number of edges.

We will show that the vertices in M are pairwise adjacent. Indeed, let v_i be the vertex with the least index for which a vertex v_j exists such that v_i, v_j are not adjacent and $j < i$. Since $d(v_i, x) = d(v_i, x_{i-1}) = 2$, in the graph G^2 the vertices $x, x_2, \ldots, x_{i-1}, v_i$ generate a simple cycle of length $i < k$. By minimality of the initial cycle, this cycle is not an induced cycle. Hence, $d(v_i, x_j) = 2$ for all $j = 1, 2, \ldots, i-1$, and the vertices of the set $A = \{B(v_j, 1) \cap M\} \cup \{v_i, x_j, x_{j+1}\}$ are pairwise adjacent at distance ≤ 2. By the Helly property, there exists a vertex v adjacent to all vertices from A, but then the set $(M\{v_j\}) \cup \{v\}$ is better than M, which contradicts the choice of the vertices in M.

Thus, the vertices $v_1, v_2, \ldots, v_{k-1}$ are pairwise adjacent. Now, pairwise intersection of balls centered at the vertices from $M \cup \{x, x_k\}$ implies existence of the vertex v_k. It remains to note that the set $\{x_1, x_2, \ldots, x_k\} \cup \{v_1, v_2, \ldots, v_k\}$ forms a subgraph of the form S_k. Since G is C_4-free, it is isometric. The contradiction proves the lemma.

Denote by $\ell(G)$ the length of the maximum induced cycle of the graph G.

LEMMA 5 [5]. For the graph G, $\ell(L(\mathfrak{N})) = \ell(L(\mathfrak{N}_1))$.

THEOREM 2. For the graph G conditions 1)-2) are equivalent:

1) G is a C_4-free HT-graph;

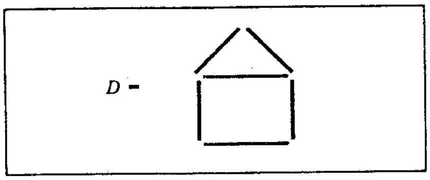

Fig. 2

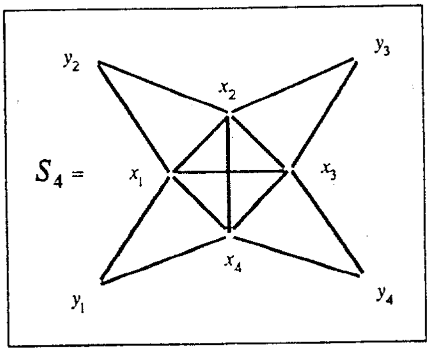

Fig. 3

2) G is a Helly graph without isometric subgraphs of the form C_4 and S_k $(k \geq 4)$.

The proof follows from Lemmas 4 and 5.

COROLLARY 2. The r-domination problem on Helly graphs without isometric subgraphs of the form C_4 and S_k $(k \geq 4)$ is solvable in polynomial time.

LEMMA 6 [20]. G is a bridged Helly graph if and only if G is a C_4- and C_5-free Helly graph.

THEOREM 3. For the graph G conditions 1)-4) are equivalent:

1) G is an HT-graph without induced subgraphs of the form S_3, C_4, C_5;

2) G is a Helly graph without isometric subgraphs of the form C_4, C_5, S_k $(k \geq 3)$;

3) G is a bridged graph without isometric subgraphs of the form S_k $(k \geq 3)$;

4) every isometric subgraph of the graph G is an HT-graph.

The proof follows from Theorem 2, Lemma 6, and Corollary 1.

COROLLARY 3. The r-domination problem on bridged graphs without isometric subgraphs of the form S_k $(k \geq 3)$ is solvable in polynomial time.

4. WEAK TRIANGULATION OF THE BALL-INTERSECTION GRAPH

LEMMA 7 [20]. Assume that the graph G satisfies the following conditions:

1) the family $\mathfrak{N}_1$ has the Helly property;

2) the graph $L(\mathfrak{N})$ is C_5-free;

3) the graph $L(\mathfrak{N}_1)$ is C_7-free.

Then G is a Helly graph.

LEMMA 8. Assume that for the graph G the graph $L(\mathfrak{N})$ is C_5-free and the graph $L(\mathfrak{N}_1)$ is $\overline{C}_k$-free $(k \geq 5)$. Then the graph $L(\mathfrak{N})$ is also $\overline{C}_k$-free $(k \geq 5)$.

Proof. We will show by induction on k $(k \geq 5)$ that the graph $L(\mathfrak{N})$ is without subgraphs of the form $\overline{C}_k$. For $k = 5$ the assertion is true because $C_5 = \overline{C}_5$. Assume that it is true for all k $(k \geq 5)$ and prove it for $k + 1$. Suppose that the balls $B(x_1, r_1)$, $B(x_2, r_2), \ldots, B(x_{k+1}, r_{k+1})$ form in the graph $\overline{L(\mathfrak{N})}$ (the complement of $L(\mathfrak{N})$) an induced cycle of length $k + 1$ with a minimum sum $\delta = r_1 + \ldots + r_{k+1}$, and $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ if and only if $i = j \pm 1 \pmod{k + 1}$. We will show that these balls are of unit radius. Assume that this is not so, i.e., $r_1 \geq 2$, say. For all vertices $y \in \langle x_1, x_3 \rangle$ and $x \in \langle x_1, x_4 \rangle$ adjacent to x_1 consider the balls $B(y, r_1 - 1)$, $B(z, r_1 - 1)$, $B(x_3, r_3)$, $B(x_4, r_4)$, $B(x_{k+1}, r_{k+1})$ that form a cycle of length 5 in $L(\mathfrak{N})$. Since the cycle is not an induced cycle and $B(x_1, r_1) \cap B(x_{k+1}, r_{k+1}) = \varnothing$, we have $B(y, r_1 - 1) \cap B(x_4, r_4) \neq \varnothing$ or $B(z, r_1 - 1) \cap B(x_3, r_3) \neq \varnothing$. Hence, at least one of the balls $B(y, r_1 - 1)$ intersects with each of the balls $B(x_4, r_4)$ and $B(x_3, r_3)$. We will show that it also intersects with all the balls $B(x_i, r_i)$, $i = 5,6,\dots,k$. Let m $(5 \leq m \leq k)$ be the least index for which the balls $B(y, r_1 - 1)$ and $B(x_m, r_m)$ are nonintersecting. Then the balls $B(y, r_1 - 1), B(x_2, r_2), \dots, B(x_m, r_m)$ form in $\overline{L(\mathfrak{N})}$ an induced cycle of length $m \leq k$. This contradicts the induction hypothesis.

Thus, the ball $B(y, r_1 - 1)$ intersects with all the balls $B(x_i, r_i)$, $i = 3,4,\dots,k$, and does not intersect with $B(x_2, r_2)$, $B(x_{k+1}, r_{k+1})$. Hence we obtain that the balls $B(y, r_1 - 1), B(x_2, r_2), \dots, B(x_k, r_k), B(x_{k+1}, r_{k+1})$ form in $\overline{L(\mathfrak{N})}$ an induced cycle of length $k + 1$ with sum of radii less than δ. Thus, all the radii of the original balls are unit radii, and therefore the graph $\overline{L(\mathfrak{N}_1)}$ contains a cycle C_{k+1} $(k \geq 5)$. This contradicts the assumption of the lemma. Q.E.D.

Recall that a graph G without subgraphs of the form C_k, $\overline{C}_k$ $(k \geq 5)$ is called weakly triangulated. Lemma 8 combined with Lemmas 5 and 7 directly leads to Theorem 4.

THEOREM 4. For the graph G the following conditions are equivalent:

1) the family $\mathfrak{N}_1$ of the graph G has the Helly property and the graph G^2 is weakly triangulated;

2) the family $\mathfrak{N}$ of the graph G has the Helly property and the graph $L(\mathfrak{N})$ is weakly triangulated.

In particular, since weakly triangulated graphs are perfect [18], the r-domination problem on Helly graphs with a weakly triangulated square is solvable in polynomial time.

THEOREM 5. If in a C_4-free Helly graph G every induced subgraph of the form $\overline{S}_3$ is suspended, then the graph G^2 is without subgraphs C_k $(k \geq 5)$ and induced houses D (Fig. 2).

Before proving the theorem, we will prove two auxiliary lemmas. In these lemmas, G is a C_4-free Helly graph.

LEMMA 9. If the vertices y_1, y_2, y_3, y_4 of the graph G form a cycle C_4 in G^2, then in G they form together with some vertices x_1, x_2, x_3, x_4 an isometric subgraph S_4 (Fig. 3).

Proof. We will first show that if the vertices y_1, y_2, y_3, y_4 of the graph G satisfy the conditions $d(y_1, y_3) = d(y_2, y_4) = 3$, $d(y_1, y_4) \leq 2$, $d(y_1, y_2) \leq 2$, $d(y_2, y_3) \leq 2$, $d(y_3, y_4) \leq 2$, then these vertices together with some vertices x_1, x_2, x_3, x_4 form in G an isometric subgraph S_4. To this end, consider the family $(B(y_1, 2), B(y_2, 2), B(y_3, 2), B(y_4, 2))$ of pairwise intersecting balls. By the Helly property there exists a vertex x_3 adjacent to y_3, y_4 for which $d(y_1, x_3) = d(y_2, x_3) = 2$. By pairwise intersection of the balls $B(y_1, 1)$, $B(y_2, 1)$, $B(x_3, 1)$ there also exists a vertex x_1 adjacent to y_1, y_2, y_3. For the same reason there exists a vertex x_2 adjacent to y_2, x_1, x_3, y_3 and a vertex x_4 adjacent to x_2, x_1, x_3, y_1, y_4. Since G is C_4-free, we have $d(y_1, y_4) = d(y_1, y_2) = d(y_2, y_3) = d(y_3, y_4) = 2$. It now remains to note that the vertices y_1, y_2, y_3, y_4 together with x_1, x_2, x_3, x_4 form an isometric subgraph S_4.

Now assume that the graph G^2 contains the cycle $C_4 = (y_1, y_2, y_3, y_4, y_1)$. Then for the vertices y_1, y_2, y_3, y_4 of the graph G we have $d(y_1, y_3) \geq 3$, $d(y_2, y_4) \geq 3$, $d(y_1, y_4) \leq 2$, $d(y_1, y_2) \leq 2$, $d(y_2, y_3) \leq 2$, $d(y_3, y_4) \leq 2$. We will show that $d(y_1, y_3) = d(y_2, y_4) = 3$, which by the above will complete the sought proof. Let, for instance, $d(y_1, y_3) = 4$. Then $d(y_i, y_{i+1}) = 2$ for all $i = 1, 2, 3, 4$ (summation is modulo 4). Consider the family $(B(y_1, 2), B(z_1, 1), B(z_2, 1))$ of pairwise intersecting balls, where $z_1 \in B(y_2, 1) \cap B(y_3, 1)$, $z_2 \in B(y_4, 1) \cap B(y_3, 1)$. By the Helly property, there exists a vertex v adjacent to z_1, z_2 which is at a distance 2 from y_1. Using the Helly property, we can also easily establish the existence of a vertex z_3 adjacent to y_1, y_2, v and a vertex z_4 adjacent to y_1, y_4, v. The cycles (y_2, z_1, v, z_3, y_2) and (y_4, z_4, v, z_2, y_4) should contain the chords (y_2, v), (y_4, v), because G is C_4-free and $d(y_1, y_3) = 4$. But then $d(y_2, y_4) = 2$, which contradicts the assumption of the lemma. Thus, $d(y_1, y_3) = 3$. We similarly prove the equality $d(y_2, y_4) = 3$.

LEMMA 10. If in the graph G every generated subgraph of the form $\overline{S}_3$ is suspended, then the graph G^2 does not contain induced houses.

Proof. Assume that G^2 contains an induced house formed by the vertices y_1, y_2, y_3, y_4, y_5, and that in G^2 the vertices y_1, y_2, y_3, y_4 generate the cycle C_4 and the vertex y_5 is adjacent to y_1 and y_2. By Lemma 9, the vertices y_1, y_2, y_3, y_4 in the graph G form together with some vertices x_1, x_2, x_3, x_4 an isometric subgraph S_4. Also in G we have the relationships $d(y_5, y_1) \leq 2$, $d(y_5, y_2) \leq 2$, $d(y_5, y_3) \geq 3$, $d(y_5, y_4) \geq 3$. In what follows, the argument is conducted for the graph G. If the vertices y_5 and x_1 are adjacent, then the set $(y_5, x_1, y_3, x_2, y_4, x_4)$ generates a subgraph $\overline{S}_3$ in G. Since $d(y_5, y_3) = d(y_5, y_4) = 3$, this subgraph is not a suspended subgraph. Now assume that the vertices y_5 and x_1 are not adjacent. By pairwise intersection of the balls $B(y_1, 1)$, $B(y_2, 1)$, $B(y_5, 1)$ there exists a vertex z adjacent to y_1, y_2, y_5. Since G is C_4-free, the vertex z is also adjacent to x_1, and so if z is also adjacent to at least one of the vertices x_2, x_4, then it is adjacent to the other one as well (otherwise a cycle C_4 would be obtained). Therefore the vertices y_5, z, y_3, x_2, y_4, x_4 generate in G a subgraph $\overline{S}_3$, which is again not a suspended subgraph. If z is adjacent to neither x_2 nor x_4, then the vertices z, x_1, y_3, x_2, y_4, x_4 induce in G a subgraph $\overline{S}_3$. Every vertex v on which $\overline{S}_3$ is suspended forms together with z, y_1, x_4 a cycle C_4.

Thus, having assumed that the graph G^2 contains an induced house, we obtain that G contains either a nonsuspended subgraph $\bar{S}_3$ or a cycle C_4. The contradictions prove the lemma. Q.E.D.

Proof of Theorem 5. It remains to show that G^2 is without cycles C_k ($k \geq 5$). The proof is by induction on the number of vertices in the graph. Since G is a Helly graph, there exist a pair of vertices $x, y \in V$ such that $B(x, 1) \subseteq B(y, 1)$. It is easy to show that the graph $G - x$ is also without cycles C_4, and any subgraph $\bar{S}_4$ is suspended in $G - x$. By the induction hypothesis, the graph $(G - x)^2$ is C_k-free ($k \geq 5$).

Assume that the graph G^2 contains a cycle C_k ($k \geq 5$) generated by the vertices $x_1, x_2, \ldots, x_k$. Without loss of generality, let $x_1 = x$. In the graph $(G - x)^2$ the vertices $y, x_2, \ldots, x_k$ form a simple cycle of length k. Since $(G - x)^2$ is C_k-free ($k \geq 5$), this cycle is divided by chords into triangles and quadrangles. Note that one of the endpoints of these chords is the vertex y. The following two cases are possible:

a) all subcycles into which the original cycle is divided by these chords are triangles;

b) at least one of the subcycles is a quadrangle.

Let us consider each case separately.

If $k \geq 6$, then the vertices x_3, x_4, x_5 of the cycle C_k are at a distance 2 from y and 3 from x in the graph G, and they satisfy in G the conditions $d(x_3, x_4) \leq 2$, $d(x_4, x_5) \leq 2$, $d(x_3, x_5) \geq 3$. By pairwise intersection of the balls $B(x_3, 1)$, $B(x_4, 1)$, $B(y, 1)$ in the graph G, there exists a vertex v adjacent to x_3, x_4, y. We similarly prove the existence of a vertex w adjacent to x_5, x_4, y. Since G is C_4-free, the vertices v, w are adjacent to one another. But then the set $\{x, y, x_3, v, x_5, w\}$ generates in G a subgraph $\bar{S}_3$, which is not suspended because $d(x, x_3) = d(x, x_5) = 3$.

Now let $k = 5$. In the graph G we have $d(x_3, y) = d(x_4, y) = 2$, $d(x_2, y) \leq 2$, $d(x_5, y) \leq 2$, $0 \neq d(x_i, x_j) \leq 2$ ($i, j = \in \{1,\ldots,3\}$). if and only if $j = i \pm 1 \pmod 5$. Since $d(x_2, x_5) \geq 3$, the vertices x_2, x_5 cannot both be adjacent to y. Thus assume that x_2 is not adjacent to y. The balls $B(x_3, 1)$, $B(x_4, 1)$, $B(y, 1)$ of the graph G are pairwise intersecting. By the Helly property, there exists a vertex v adjacent to y, x_3, x_4. Similarly there exists a vertex v adjacent to y, x_2, x_3. Since G is C_4-free, the vertices u and v are adjacent to one another. For the same reason the vertices x_2 and x are nonadjacent. Then the set $\{x, y, x_2, v, x_4, u\}$ generates in G a subgraph $\bar{S}_3$, which is not suspended because $d(x_2, x_4) = d(x, x_4) = 3$.

b) All the chords of the cycle $(y, x_2, \ldots, x_k, y)$ in $(G - x)^2$, as noted above, originate from the vertex y, and therefore the quadrangle, being a subcycle of this cycle, has the form $(y, x_i, x_{i+1}, x_{i+2}, y)$. If $i = 2$ (or $i + 2 = k$), then the vertices x, y, x_{i+1}, x_{i+2} generate a house in G^2, a contradiction with Lemma 10. Assume that $i \neq 2$ and $i + 2 \neq k$. Then $d(x, x_i) \geq 3$ and $d(x, x_{i+2}) \geq 3$. By Lemma 9, the vertices x_i, y, x_{i+1}, x_{i+2} together with some vertices z_1, z_2, z_3, z_4 form an isometric subgraph S_4. Let $z_1 \in \langle y, x_i \rangle$, $z_2 \in \langle y, x_{i+2} \rangle$. It is easy to note that the vertices x, y, x_i, z_1, x_{i+2}, z_2 generate in G a subgraph $\bar{S}_3$, and since $d(x, x_i) = d(x, x_{i+2}) = d(x_i, x_{i+2}) = 3$ the subgraph $\bar{S}_3$ is not suspended in G.

The contradictions prove the theorem. Q.E.D.

Since every graph $\bar{C}_k$ ($k \geq 6$) contains a house as an induced subgraph, Theorems 4 and 5 lead to the following corollary.

COROLLARY 4. The r-domination problem on C_4-free Helly graphs in which every generated subgraph $\bar{S}_3$ is suspended is solvable in polynomial time.

COROLLARY 5. The r-domination problem on bridged graphs without induced subgraphs of the form S_3 and $\bar{S}_3$ is solvable in polynomial time.

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