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LexBFS-orderings of distance-hereditary graphs with application to the diametral pair problem ☆

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Abstract

For an undirected graph G the kth power G^k of G is the graph with the same vertex set as G where two vertices are adjacent iff their distance is at most k in G. In this paper we prove that every LexBFS-ordering of a distance-hereditary graph is both a common perfect elimination ordering of all even powers and a common semi-simplicial ordering of all powers of this graph. Moreover, we characterize those distance-hereditary graphs by forbidden subgraphs for which every LexBFS-ordering of the graph is a common perfect elimination ordering of all powers. As an application we present an algorithm which computes the diameter and a diametral pair of vertices of a distance-hereditary graph in linear time. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Distance-hereditary graphs; LexBFS-ordering; Perfect elimination ordering; Metric power; Diameter; Linear-time algorithm

1. Introduction

In recent years some papers investigating powers of certain graphs were published. One of the first results in this field is due to Duchet [10]: If G^k is chordal then G^{k+2} is so. In particular, odd powers of chordal graphs are chordal, whereas even powers of chordal graphs are in general not chordal. In [1] it was shown that even powers of distance-hereditary graphs are chordal and odd powers do not contain a house, hole or domino as induced subgraph, i.e. they are HHD-free.

It is well known that every chordal graph has a perfect elimination ordering. Thus each chordal power of an arbitrary graph has a perfect elimination ordering. A natural question is whether there is a common perfect elimination ordering of all (or

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some) chordal powers of a given graph. The first result in this direction using minimal separators is given in [9]: If both G and G^2 are chordal then there is a common perfect elimination ordering of these graphs. The existence of a common perfect elimination ordering of all chordal powers of an arbitrary given graph was proved in [4]. Such a common ordering can be computed in time $O(|V||E|)$ using a generalized version of Maximum Cardinality Search whichsimultaneously uses chordality of these powers.

As shown in [18] lexicographic breadth-first-search (LexBFS) gives a perfect elimination ordering of a chordal graph in linear time. In [5] we proved that every LexBFS-ordering of a chordal graph G gives a common perfect elimination ordering of all odd powers of G. Moreover, we characterized those chordal graphs by forbidden isometric subgraphs for which every LexBFS-ordering of the graph is a common perfect elimination ordering of all powers.

In this paper we consider LexBFS-orderings of distance-hereditary graphs. Since distance-hereditary graphs are House-Hole-Domino-free each LexBFS-ordering of G is a semi-simplicial ordering of G [14,17]. We will prove that every LexBFS-ordering of a distance-hereditary graph is both a common perfect elimination ordering of all even powers and a common semi-simplicial ordering of all powers of this graph. In [15] a tree structure for distance-hereditary graphs was given, i.e. distance-hereditary graphs were characterized as the graphs for which the family of all maximal connected cographs is a dual hypertree. It turns out that the dismantling scheme corresponding to this tree structure is a perfect elimination ordering of the square of the graph. Thus our results give another linear time method for computing such a dismantling scheme by only considering the graph itself.

Furthermore, we characterize those distance-hereditary graphs by forbidden subgraphs for which every LexBFS-ordering of the graph is a common perfect elimination ordering of all powers. For such graphs every LexBFS-ordering $\sigma = (v_1, \ldots, v_n)$ is a 'diametral' ordering too, i.e. each vertex v_i is diametral in the subgraph induced by $\{v_i, \ldots, v_n\}$. Such a special ordering is used in [16] for solving the problems Hamiltonian circuit and path efficiently for distance-hereditary graphs. For general distance-hereditary graphs only a quadratic time for computing such an ordering is known. But in the case of ptolemaic graphs (those graphs which are both chordal and distance-hereditary) our result leads to a linear time algorithm. Thus the Hamiltonian problems can be solved for ptolemaic graphs in linear time (cf. [16]).

Finally, as an application of the results of the first part of this paper, we present a simple algorithm which computes both the diameter and a diametral pair of vertices of a distance-hereditary graph in linear time. Note that in [7] a linear time algorithm for computing the diameter of a distance-hereditary graph was presented, but that approach is not usable for finding a diametral pair of vertices.

Computing the diameter of a graph or a pair of diametral vertices is important in network design. For example, by considering the routing problem in a network it is obvious that for each single-source routing algorithm (i.e. given a vertex as source one has to reach all other vertices of the network) the lower bound of its running time is the diameter of the network. On the other hand, distance-hereditary graphs seem well suited for high reliable networks. Taking an k-connected distance-hereditary graph ($k\geqslant 2$) as network the removal of at most $k-1$ vertices (which corresponds to the failure of processors) does not change any distances (and thus routing times) within the network.

All these results show the usefulness of LexBFS-orderings not only for generating special dismantling schemes of graphs, but also for solving certain optimization problems.

2. Preliminaries

Throughout this paper all graphs $G=(V,E)$ are finite, undirected, simple (i.e. loop-free and without multiple edges) and connected.

A *path* is a sequence of vertices $v_0,\dots,v_k$ such that $v_iv_{i+1}\in E$ for $i=0,\dots,k-1$; its *length* is k. As usual, an induced path of k vertices is denoted by P_k. A graph G is *connected* iff for every pair of vertices of G there is a path in G joining both vertices.

The *distance* $d_G(u,v)$ of vertices u,v is the minimal length of a path connecting these vertices. If no confusion can arise we will omit the index G.

The *eccentricity* $e(v)$ of a vertex $v\in V$ is the maximum over $d(v,x)$, $x\in V$. The minimum over the eccentricities of all vertices of G is the *radius* $rad(G)$ of G, whereas the maximum is the *diameter* $diam(G)$ of G. A pair x,y of vertices of G is called *diametral* iff $d(x,y)=diam(G)$.

The *kth neighbourhood* $N^k(v)$ of a vertex v of G is the set of all vertices of distance k to v, i.e.

$$N^k(v):=\{u\in V\colon d_G(u,v)=k\},$$

whereas the *disk* of radius k centered at v is the set of all vertices of distance at most k to v:

$$D_G(v,k):=\{u\in V\colon d_G(u,v)\leqslant k\}=\bigcup_{i=0}^{k}N^i(v).$$

For convenience we will write $N(v)$ instead of $N^1(v)$. Again, if no confusion can arise we will omit the index G. The *kth power* G^k of G is the graph with the same vertex set V where two vertices are adjacent iff their distance is at most k.

Next we recall the definition and some characterizations of the considered graph classes. An *induced cycle* is a sequence of vertices $v_0,\dots,v_k$ such that $v_0=v_k$ and $v_iv_j\in E$ iff $|i-j|=1\,(\text{modulo}\,k)$. The *length* $|C|$ of a cycle C is its number of vertices. In the sequel a *hole* is an induced cycle of length at least five. A graph G is *chordal* iff every induced cycle of G is of length three. One of the first results on chordal graphs is the characterization via dismantling schemes. A vertex v of G is called *simplicial* iff $D(v,1)$ induces a complete subgraph of G. A *perfect elimination* *ordering* is an ordering of G such that v_i is simplicial in $G_i := G(\{v_i,\dots,v_n\})$ for each $i = 1,\dots,n$. It is well known that a graph is chordal if and only if it has a perfect elimination ordering (cf. [11]). Moreover, there is a linear time algorithm for computing perfect elimination orderings of chordal graphs: Lexicographic breadth-first-search (LexBFS, [11]). To make the paper self-contained we present the rules of this algorithm. LexBFS orders the vertices of a graph by assigning numbers from $n = |V|$ to 1 in the following way: assign the number k to a vertex v (as yet unnumbered) which has lexically largest vector (s_i: $i = n, n-1,\dots,k+1$), where $s_i = 1$ if v is adjacent to the vertex numbered i, and $s_i = 0$ otherwise. An ordering $\sigma = (v_1, v_2,\dots,v_n)$ of the vertex set of a graph G generated by LexBFS will be called *LexBFS-ordering* of G.

In what follows we will often use the following property of LexBFS-orderings (cf. [14]):

(P1) If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $db \in E$ and $da \notin E$.

We write $x < y$ whenever in a given ordering σ vertex x has a smaller number than vertex y. Moreover, $x < \{y_1,\dots,y_k\}$ is an abbreviation for $x < y_i$, $i = 1,\dots,k$. It is well known that any LexBFS-ordering has property (P1) [11]. Moreover, any ordering fulfilling (P1) can be generated by LexBFS [5].

An induced subgraph H of G is an *isometric* subgraph of G iff the distances within H are the same as in G, i.e.

$$\forall x, y \in V(H)\colon d_H(x, y) = d_G(x, y).$$

A graph G is *distance-hereditary* [13] iff each connected induced subgraph of G is isometric. Distance-hereditary graphs were extensively studied in [2,12,6,1,15]. For proving our results we will often use the following property:

Theorem 2.1 (The four point condition [2]). *Let G be a distance-hereditary graph. Then, for every four vertices u, v, w, x at least two of the distance sums*

$$d(u,v) + d(w,x), \quad d(u,w) + d(v,x), \quad d(u,x) + d(w,v)$$

are equal, and, if the two smaller sums are equal then the larger one exceeds this by at most two.

Furthermore, distance-hereditary graphs can be characterized by forbidden subgraphs [2,12]: A graph is distance-hereditary if and only if it does not contain a hole, a house, a domino and a 3-fan as induced subgraph (see Fig. 1).

Finally, a graph G is called *pseudo-modular* [3] iff for every three vertices x_1, x_2, x_3 of G there are vertices z_1, z_2, z_3 of G such that

$$d(x_i, x_j) = d(x_i, z_i) + d(z_i, z_j) + d(z_j, x_j), \quad i \neq j \in \{1, 2, 3\}$$

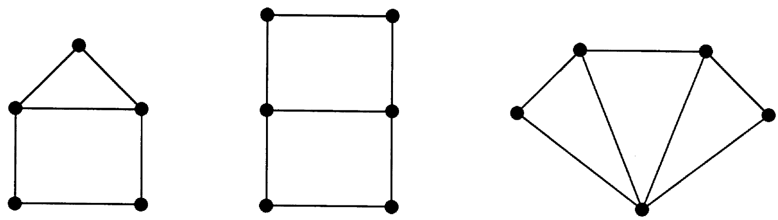


Fig. 1. A house, a domino and a 3-fan.

and $z_1 = z_2 = z_3$ or $\{z_1, z_2, z_3\}$ is complete. In [3] it was shown that distance-hereditary graphs are pseudo-modular.

3. LexBFS-orderings and powers of distance-hereditary graphs

By definition, HHD-free graphs are the graphs which do not contain a hole, a house or a domino as induced subgraphs. These graphs were introduced in [14] (see also [17]) as the graphs for which every LexBFS-ordering of every induced subgraph is a semi-simplicial ordering. Hereby, a vertex is *semi-simplicial* iff it is not a midpoint of a P_4. A *semi-simplicial ordering* $(v_1, \dots, v_n)$ is an ordering of the vertex set of G such that v_i is semi-simplicial in $G_i := G(\{v_i, \dots, v_n\})$.

In [8] we characterized HHD-free graphs by means of convexity. A subset S of V is called *m^3-convex* iff S contains every induced path of length at least three between vertices of S. Note that m^3-convexity is a relaxation of m-convexity which is useful for characterizing chordal graphs.

Theorem 3.1 (Dragan et al. [8]). *The following conditions are equivalent for a graph G*:
(1) *G is HHD-free.*
(2) *Every LexBFS-ordering of G is a semi-simplicial ordering.*
(3) *For every LexBFS-ordering $(v_1, \dots, v_n)$ of G the set $V(G_i)$ is m^3-convex in G for all $i = 1, \dots, n$.*
(4) *The disks $D(v, k)$, $k \geqslant 1$, are m^3-convex for all vertices $v \in V$.*

For proving the results we will frequently use the following corollary of condition (3) of the above theorem:

Corollary 3.2. *If σ is a LexBFS-ordering of G and $u_1 - \cdots - u_k$ is an induced path of length $k \geqslant 3$ then either u_1 or u_k must be the leftmost vertex of the path with respect to σ.*

The definition of semi-simplicial vertices immediately implies

Remark 3.3. For each semi-simplicial vertex v of a graph G such that $e(v) \geqslant 2$, the subgraph $G \backslash \{v\}$ is isometric in G.

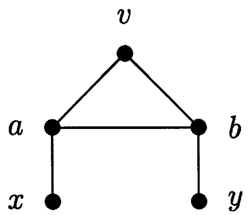
which forms the basis of the inductive proofs in the sequel.

3.1. Even powers of distance-hereditary graphs

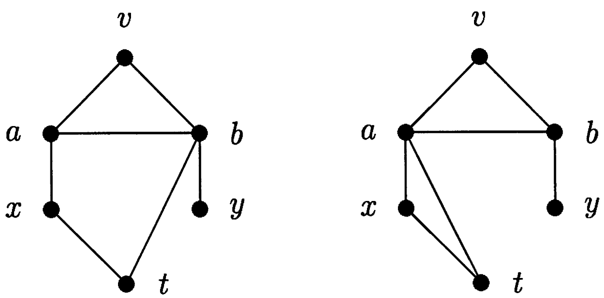
In [1] it was proved that all even powers of a distance-hereditary graph are chordal. In this section we show that any LexBFS-ordering of a given distance-hereditary graph is a common perfect elimination ordering of all its even powers.

Lemma 3.4. *Each LexBFS-ordering σ of a distance-hereditary graph G is a perfect elimination ordering of G^2.*

Proof. Let v be the first vertex of σ and assume that there are vertices $x, y \in D(v, 2)$ such that $d(x, y) \geqslant 3$. Since distance-hereditary graphs are HHD-free, v is semi-simplicial in G, thus $d(x, y) = 3$ and $x, y \in N^2(v)$. Let a, b be adjacent vertices of $N(v)$ such that $ax \in E$ and $by \in E$. W.l.o.g. we may assume $a < b$. Now applying m^3-convexity to the induced path $x - a - b - y$ gives $\min\{x, y\} < a$.

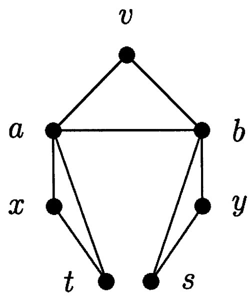


Case 1: $x < a < y$. We immediately conclude $x < b$. Applying (P1) to $v < x < b$ yields a vertex $t > b$ adjacent to x but not to v. Since x is the smallest vertex of the path $t - x - a - b$ this path cannot be induced by m^3-convexity. Thus either $ta \in E$ or $tb \in E$. Assuming $tb \in E$ gives either a 3-fan (for $ta \in E$) or a house (for $ta \notin E$), a contradiction in both cases. Therefore $tb \notin E$ and $ta \in E$. But now the path $t - a - b - y$ is induced and a is its smallest vertex, a contradiction to m^3-convexity.



Case 2: $y < a < x$. Analogously to Case 1, applying (P1) to $v < y < a$ gives a vertex $s > a$ adjacent to y and b but neither to v nor to a. Thus the path $s - b - a - x$

is induced and its minimum vertex is a, a contradiction to m^3-convexity.



Case 3: $x, y < a$. Since $x < a$ and $y < a$ we get vertices t and s as described in Cases 1 and 2, respectively. Now the minimum vertex of the path $t - a - b - s$ is a. By m^3-convexity this path cannot be induced, implying $st \in E$. But now $\{s, t, a, b, v\}$ induces a house.

Induction and Remark 3.3 settle the proof. □

Theorem 3.5. *Each LexBFS-ordering σ of a distance-hereditary graph G is a perfect elimination ordering of each even power G^{2k}, $k \geqslant 1$.*

Proof. By Remark 3.3 it is sufficient to prove that the first vertex of a LexBFS-ordering is simplicial in G^{2k}. This we show by induction on k.

For $k = 1$ we are done by Lemma 3.4. So we may assume $k \geqslant 2$. Let v be the first vertex of σ and assume that there are vertices $x, y \in D(v, 2k)$ such that $d(x, y) \geqslant 2k+1$. By the induction hypothesis v is simplicial in the even powers $G^2, \ldots, G^{2k-2}$. In particular, the distance between any two vertices of $D(v, 2k-2)$ is at most $2k-2$. We conclude $x, y \notin D(v, 2k-2)$ and $d(x, y) \leqslant 2k+2$. Moreover, both vertices x and y cannot be in $N^{2k-1}(v)$, since otherwise $d(x, y) \leqslant 2k$ will hold.

Case 1: $x \in N^{2k}(v)$ and $y \in N^{2k-1}(v)$. Choose vertices $a_1, b \in N^{2k-2}(v)$ and $a_2 \in N^{2k-1}(v)$ rightmost in σ such that $a_1 - a_2 - x$ induces a P_3 and $yb \in E$. We immediately obtain the following equalities:

$$d(a_1, b) = 2k - 2, \quad d(a_2, b) = d(a_1, y) = 2k - 1,$$
$$d(a_2, y) = 2k, \quad d(x, y) = 2k + 1.$$

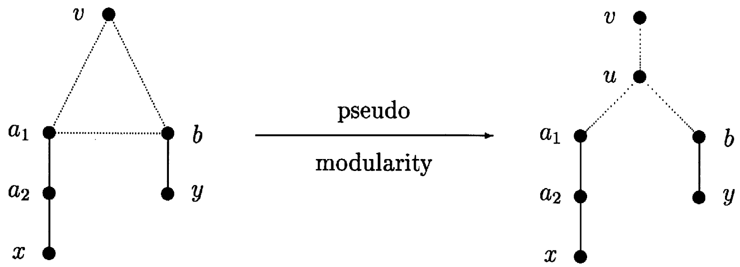
By pseudo-modularity of G there is a vertex u such that

$$d(v, u) = d(u, a_1) = d(u, b) = k - 1.$$

Let z be the neighbour of v on a shortest path joining v and u and i its position in σ. We obtain

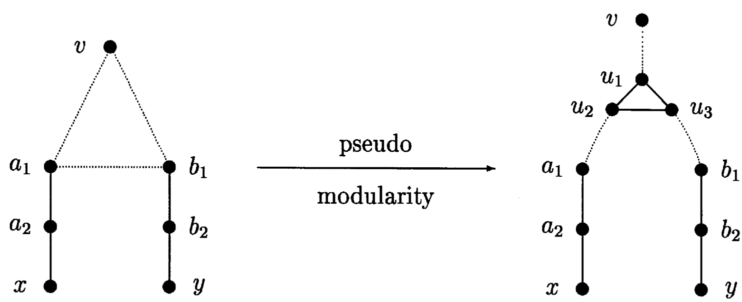
$$d(z, a_1) = d(z, b) = 2k - 3 \quad \text{and} \quad d(z, a_2) = d(z, y) = 2k - 2.$$

Since $d(a_2, y) = 2k$, $a_2, y \in D(z, 2k-2)$ and z is simplicial in G_i^{2k-2} not both vertices a_2, y can be contained in G_i, i.e. $\min\{a_2, y\} < z$.



Case 1.1: $y < z < a_2$. Applying (P1) to $v < y < z$ gives a vertex $t > z$ adjacent to y but not to v. Let $b - w_1 - \cdots - w_l - z$ be a shortest path joining b and z and define $P := t - y - b - w_1 - \cdots - w_l - z$. Since $|P| \geqslant 3$ and y is smaller than t and z the path P cannot be induced by m^3-convexity, i.e. $tb \in E$ or $tw_1 \in E$. We conclude $2k-3 \leqslant d(z,t) \leqslant 2k-2$. Thus both vertices a_2 and t are in $D(z, 2k-2)$ in G_i. But $d(a_2, y) = 2k$ and $yt \in E$ imply $d(a_2, t) \geqslant 2k-1$, a contradiction to the simpliciality of z in G_i^{2k-2}.

Case 1.2: $a_2 < z$. Let $a_1 - w_1 - \cdots - w_l - z$ be a shortest path joining a_1 and z. Since $a_2 < z$ and the length of the induced path $x - a_2 - a_1 - w_1 - \cdots - w_l - z$ is at least three m^3-convexity implies $x < a_2$. Now applying (P1) to $v < x < z$ gives a vertex $t > z$ adjacent to x but not to v. Since x is smaller than t and z the path $t - x - a_2 - a_1 - w_1 - \cdots - w_l - z$ cannot be induced by m^3-convexity. Hence, by distance requirements $ta_1 \in E$ or $ta_2 \in E$. If $ta_1 \notin E$ then $ta_2 \in E$ and the path $t - a_2 - a_1 - w_1 - \cdots - w_l - z$ is induced. This implies a contradiction to m^3-convexity since a_2 is smaller than t and z. Therefore $ta_1 \in E$. But now we can replace a_2 by $t > a_2$, a contradiction to the choice of a_2.



Case 2: $x, y \in N^{2k}(v)$. Choose vertices $a_1, b_1 \in N^{2k-2}(v)$ and $a_2, b_2 \in N^{2k-1}(v)$ such that $a_1 - a_2 - x$ and $b_1 - b_2 - y$ are induced paths. Since we already handled Case 1 we may assume $d(a_2, y), d(b_2, x) \neq 2k+1$. We immediately conclude

$$d(a_2, y) = d(b_2, x) = 2k \quad \text{and} \quad d(x, y) = 2k+1.$$

Now consider the distance sums of the vertices v, x, y, a_1:

$$d(v,x)+d(a_1,y)=2k+d(a_1,y)\geqslant 4k-1,$$
$$d(v,a_1)+d(x,y)=4k-1,$$
$$d(v,y)+d(a_1,x)=2k+2.$$

The four-point condition now implies

$$d(v,x)+d(a_1,y)=d(v,a_1)+d(x,y)=4k-1.$$

Thus $d(a_1,y)=2k-1$ and by symmetry $d(b_1,x)=2k-1$. So we can compute the distance sums for x,y,a_1,b_1:

$$d(a_1,x)+d(b_1,y)=4,$$
$$d(a_1,b_1)+d(x,y)=2k+1+d(a_1,b_1)\geqslant 4k-2,$$
$$d(a_1,y)+d(b_1,x)=4k-2.$$

The four-point condition gives

$$d(a_1,b_1)+d(x,y)=d(a_1,y)+d(b_1,x)=4k-2$$

and thus $d(a_1,b_1)=2k-3$. This immediately implies $d(a_2,b_2)=2k-1$. From $d(a_1,b_1)=2k-3$ and $d(v,a_1)=d(v,b_1)=2k-2$ pseudo-modularity yields pairwise adjacent vertices u_1,u_2,u_3 such that

$$d(v,u_1)=k-1 \quad \text{and} \quad d(a_1,u_2)=d(b_1,u_3)=k-2.$$

Let z be the neighbour of v on a shortest path joining v and u_1 and i its position in σ. Then

$$d(z,a_1)=d(z,b_1)=2k-3 \quad \text{and} \quad d(z,a_2)=d(z,b_2)=2k-2.$$

Therefore the vertices a_2,b_2 are contained in $D(z,2k-2)$. By induction hypothesis z is simplicial in G_i^{2k-2}. Thus $d(a_2,b_2)=2k-1$ implies $\min\{a_2,b_2\}<z$. Due to symmetry we may assume $a_2<z$. Let $a_1-w_1-\cdots-w_l-z$ be a shortest path joining a_1 and z. Since $a_2<z$ and the length of the induced path $x-a_2-a_1-w_1-\cdots-w_l-z$ is at least three m^3-convexity implies $x<a_2$. Now (P1) applied to $v<x<z$ yields a vertex $t>z$ adjacent to x but not to v (note $t\neq a_2$ since $a_2<z<t$). Since vertex x is smaller than the endvertices t,z of the path $t-x-a_2-a_1-w_1-\cdots-w_l-z$ we conclude $ta_2\in E$ or $ta_1\in E$ by m^3-convexity. If $ta_1\notin E$ then $ta_2\in E$ and the path $t-a_2-a_1-w_1-\cdots-w_l-z$ is induced. But a_2 is smaller than t and z, a contradiction to the m^3-convexity. Therefore $ta_1\in E$ which immediately implies $d(z,t)=2k-2$ and $d(t,b_2)=2k-1$.

If $b_2>z$ then both vertices b_2 and t are contained in $D(z,2k-2)$ in G_i, a contradiction to the simpliciality of z in G_i^{2k-2}. If $b_2<z$ then we can apply the same arguments as above to b_2,b_1,y yielding a vertex $s>z$ adjacent to y and b_1 with $d(z,s)=2k-2$. From $tx\in E$, $sy\in E$ and $d(x,y)=2k+1$ we conclude $d(s,t)\geqslant 2k-1$. But both s and t are contained in $D(z,2k-2)$ in G_i, again a contradiction to the simpliciality of z in G_i^{2k-2}. □

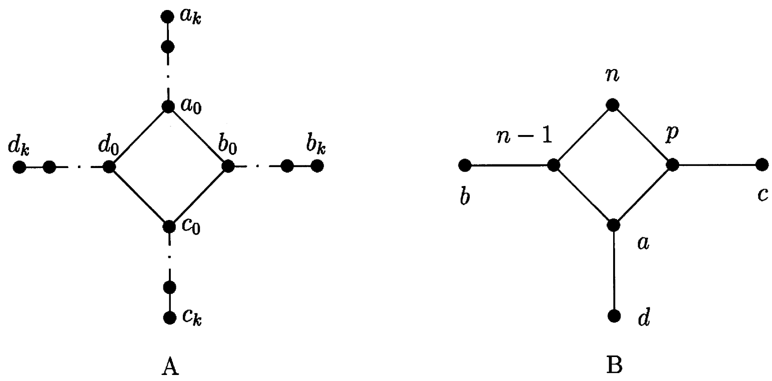


Fig. 2. A $C_4^{(k)}$ and a $C_4^{(1)}$ minus a pendant vertex.

3.2. Odd powers of distance-hereditary graphs

In [1] it was proved that all odd powers of a distance-hereditary graph are HHD-free. Moreover, an odd power G^{2k+1} is chordal if and only if G does not contain an induced subgraph isomorphic to the $C_4^{(k)}$, cf. Fig. 2A.

Theorem 3.6. *Each LexBFS-ordering σ of a given distance-hereditary graph G is a common perfect elimination ordering of all its powers if and only if G does not contain the graph of Fig.* 2B *as induced subgraph.*

Proof. Assume that a distance-hereditary graph G contains the graph of Fig. 2B as induced subgraph. We will construct a LexBFS-ordering σ of G which is not a perfect elimination ordering of its cube. We start the LexBFS-procedure at the vertex labeled by n. Let the labels of the graph of Fig. 2B be given by σ. Obviously, $\{a,d\} < p < n-1 < n$ and $c < b$. Suppose $a < b$ and let i be the position of a in σ. Then a is not simplicial in $(G_i)^2$ since $a < \{b,p\}$, $b,p \in D(a,2)$ and $d(b,p) = 3$. Thus $c < b < a < p < n-1 < n$. Now we will show $d < c$ which implies that d is not simplicial in $(G_j)^3$ where j is the position of d in σ. Assume $c < d$. Applying (P1) to $c < d < p$ gives a vertex $t > p$ adjacent to d but not to c. Since $t > p$ we have $t \neq a$, and $tn \in E$ by the rules of LexBFS. But this is a contradiction to $d(n,d) = 3$.

To prove the converse let G be a distance-hereditary graph which does not contain the graph of Fig. 2B. By Theorem 3.5 it is sufficient to show by induction on k that σ is a perfect elimination ordering for G^{2k+1}, $k \geqslant 1$. By Remark 3.3 we have only to prove that the first vertex v of σ is simplicial in G^{2k+1}. We start with the cube of G. Assume that there are vertices $x,y \in D(v,3)$ such that $d(x,y) \geqslant 4$. Since v is simplicial in G^2 by Theorem 3.5 we immediately have

$$d(v,x) = d(v,y) = 3 \quad \text{and} \quad d(x,y) = 4.$$

Choose vertices $a, b \in N^2(v)$ rightmost in σ such that $ax \in E$ and $by \in E$. Since v is simplicial in G^2 and $d(x,y) = 4$ we must have

$$d(a,b) = 2 \quad \text{and} \quad d(a,y) = d(b,x) = 3.$$

Thus, by pseudo-modularity there is a vertex $u \in N(v)$ adjacent to v, a, b. We may choose u rightmost in σ. Let i be the position of u in σ. W.l.o.g. we may assume $a < b$. Since the path $x - a - u - b$ is induced and $a < b$ m^3-convexity gives $x < a$. On the other hand, both vertices a and y are contained in $D(u,2)$. Since $d(a,y) = 3$ and u is simplicial in G_i^2 we must have $a < u$ or $y < u$. Assuming $a < u$ immediately gives $x < u$. Thus (P1) applied to $v < x < u$ yields a vertex $t > u$ adjacent to x but not to v. Note $t \neq a$. Now vertices x and a are smaller than t, u and b. Thus m^3-convexity with respect to the path $t - x - a - u - b$ implies $tu \in E$. Therefore, we can replace a by $t > a$, a contradiction to the choice of a. Consequently, $y < u < a$ holds. Property (P1) applied to $u < a < b$ yields a vertex $t > b$ adjacent to a but not to u. Since u is smaller than the endvertices t, b of the path $t - a - u - b$ and $tu \notin E$ we infer $tb \in E$ from m^3-convexity. By distance requirements t cannot be adjacent to x and y. If $tv \in E$ then we can replace u by $t > u$, a contradiction to the choice of u. Thus $tv \notin E$ and we have constructed an induced subgraph isomorphic to the graph of Fig. 2B, a contradiction. Consequently, v is simplicial in G^3.

Now let $k \geqslant 2$ and assume that there are vertices x, y in $D(v, 2k+1)$ such that $d(x,y) \geqslant 2k+2$. Since v is simplicial in G^{2k} by Theorem 3.5 there must be vertices $a, b \in N^{2k}(v)$ such that

$$d(a,b) = 2k, \quad d(v,x) = d(v,y) = 2k+1, \quad d(x,y) = 2k+2 \quad \text{and} \quad ax, by \in E.$$

We may choose a and b rightmost in σ and assume $a < b$ by symmetry. Now we have $d(v,a) = d(v,b) = d(a,b) = 2k$ and thus pseudo-modularity gives a vertex u such that

$$d(v,u) = d(a,u) = d(b,u) = k.$$

Let z be the neighbour of v on a shortest path joining v and u and i its position in σ. By the induction hypothesis z is simplicial in G^{2k-1}. Since $a, b \in D(z, 2k-1)$ but $d(a,b) = 2k$ we must have $a < z$. Let $a - w_1 - \cdots - w_l - z$ be a shortest path joining a and z. Since $a < z$ m^3-convexity with respect to $x - a - w_1 - \cdots - w_l - z$ yields $x < a < z$. Applying (P1) to $v < x < z$ gives a vertex $t > z$ adjacent to x but not to v. Note $t \neq a$. Now the path $t - x - a - w_1 - \cdots - w_l - z$ cannot be induced by m^3-convexity. By distance requirements the only possible chords are ta and tw_1. If $tw_1 \notin E$ then $ta \in E$ and $t - a - w_1 - \cdots - w_l - z$ is induced. But a is smaller than t and z, a contradiction to m^3-convexity. Thus $tw_1 \in E$ and we can replace a by $t > a$, a contradiction to the choice of a. □

Finally, using the four point condition and Theorem 3.5 we show

Theorem 3.7. *Every LexBFS-ordering σ of a distance-hereditary graph G is a common semi-simplicial ordering of all its powers.*

Proof. Since each distance-hereditary graph is HHD-free and each perfect elimination ordering is a semi-simplicial ordering the result holds for G and all even powers. Again, by Remark 3.3 it is sufficient to consider the first vertex v of σ. Assume that v is a midpoint of a P_4 $x-v-y-w$ in G^{2k+1}, $k \geqslant 1$. Since v is simplicial in G^{2k} we have

$$d(a,b)=2k, \quad d(v,x)=d(v,y)=2k+1, \quad d(x,y)=2k+2 \quad \text{and} \quad ax, by \in E$$

for some vertices $a,b \in N^{2k}(v)$. By pseudo-modularity there is a vertex z such that

$$d(v,z)=d(a,z)=d(b,z)=k.$$

To apply the four-point condition we compute the distance sums of the vertices v,y,z,w:

$$d(v,z)+d(y,w)=k+d(y,w) \leqslant 3k+1,$$

$$d(v,y)+d(z,w)=2k+1+d(z,w)>3k+2,$$

$$d(v,w)+d(z,y)=k+1+d(v,w)>3k+2.$$

Thus the second and third sum must be equal giving $d(z,w)+k=d(v,w)$. Since $d(z,v)=k$ we obtain $d(v,w)=d(v,z)+d(z,w)$, i.e. z lies on a shortest path joining v and w. We proceed by considering x,y,z,w:

$$d(x,z)+d(y,w)=k+1+d(y,w) \leqslant 3k+2,$$

$$d(x,w)+d(z,y)=k+1+d(x,w)>3k+2,$$

$$d(x,y)+d(z,w)=2k+2+d(z,w)>3k+3.$$

Thus the second and third sum must be equal giving $d(x,w)=k+1+d(z,w)$. Since $d(z,x)=k+1$ we obtain $d(x,w)=d(x,z)+d(z,w)$, i.e. z lies on a shortest path joining x and w.

Let c be a vertex in $N^{2k+2}(v)$ which lies on a shortest path joining z and w, i.e. $d(z,w)=d(z,c)+d(c,w)$. By Theorem 3.5 vertex v is simplicial in G^{2k+2}. Thus, from $c,x \in D(v,2k+2)$ we conclude $d(x,c) \leqslant 2k+2$. Now we obtain a contradiction by the following inequalities:

$$d(x,w) \leqslant d(x,c)+d(c,w) \leqslant 2k+2+d(c,w),$$

$$d(x,w)=d(x,z)+d(z,w)=d(x,z)+d(z,c)+d(c,w)=2k+3+d(c,w)$$

which is impossible. □

4. Computing a diametral pair of vertices

In this section we apply the preceding results to compute the diameter and a diametral pair of vertices of a distance-hereditary graph in linear time.

Lemma 4.1. *Let v be the first vertex of a LexBFS-ordering of a distance-hereditary graph G. Then*

$$diam(G) - 1 \leqslant e(v) \leqslant diam(G).$$

Moreover, if $e(v)$ is even then $e(v) = diam(G)$.

Proof. If $e(v)=2k$, $k \geqslant 1$, then G^{2k} is complete by Theorem 3.5, and thus $diam(G)=2k$. If $e(v) = 2k+1$, $k \geqslant 1$, then G^{2k+2} is complete by Theorem 3.5, and hence $2k+1 \leqslant diam(G) \leqslant 2k+2$. □

Corollary 4.2. *Let G be a distance-hereditary graph which does not contain the graph of Fig.* 2B *as induced subgraph, and let v be the first vertex of a LexBFS-ordering of G. Then $e(v) = diam(G)$.*

Proof. Immediately follows from Theorem 3.6 and the proof of Lemma 4.1. □

For the sequel we may assume that G is not complete for otherwise there is nothing to do. In what follows we describe the steps of the algorithm.

At first we compute a LexBFS-ordering σ of a given distance-hereditary graph G. Let v be the first vertex of σ. If $e(v)=2k$, $k \geqslant 1$, then, by Lemma 4.1, $e(v)=diam(G)$, and the vertices v and $w \in N^{e(v)}(v)$ form a diametral pair of G. So let $e(v) = 2k+1$. Now we start LexBFS at vertex v yielding a LexBFS-ordering τ with first vertex u. If $e(u)=2k+2$ then, by Lemma 4.1, $diam(G)=2k+2$ and the vertices u and $w \in N^{e(u)}(u)$ form a diametral pair of G. Otherwise ($e(v) = e(u) = 2k+1$) we choose a vertex z at distance k to u and at distance $k+1$ to v.

Lemma 4.3. $k+1 \leqslant e(z) \leqslant k+2$.

Proof. Since $d(z,v) = k+1$ we immediately have $e(z) \geqslant k+1$. So let w be a vertex of V such that $d(z,w) \geqslant k+2$. We obtain the following distance sums:

$$d(u,v) + d(z,w) = 2k+1+d(z,w) \geqslant 3k+3,$$

$$d(u,z) + d(v,w) = k + d(v,w) \leqslant 3k+1,$$

$$d(u,w) + d(v,z) = k+1+d(u,w) \leqslant 3k+2.$$

Now the four-point condition gives

$$d(v,w) = 2k+1, \quad d(u,w) = 2k \quad \text{and} \quad d(z,w) = k+2.$$

This settles the proof. □

For every vertex w of $V \backslash D(z,k)$ we store in $track(w)$ the second edge of an arbitrary shortest path from z to w. Define $F := \{track(w)\colon w \in V \backslash D(z,k)\}$. We will say that two edges in a graph are *independent* iff the vertices of this edges induce a $2K_2$ in G.

Lemma 4.4. *$diam(G) = 2k + 2$ if and only if the set F contains two independent edges.*

Proof. Let $diam(G) = 2k + 2$ and let x, y be vertices of G such that $d(x, y) = 2k + 2$. Since both u and v (as first vertices of LexBFS-orderings) are simplicial in G^{2k} we get

$$d(u,x) = d(u, y) = d(v,x) = d(v, y) = 2k + 1.$$

With $d(z,u) = k$ this implies $d(z,x) \geqslant k + 1$. So we obtain the following distance sums:

$$d(u, v) + d(z,x) = 2k + 1 + d(z,x) \geqslant 3k + 2,$$

$$d(u,z) + d(v,x) = k + 2k + 1 = 3k + 1,$$

$$d(u,x) + d(v,z) = 2k + 1 + k + 1 = 3k + 2.$$

Now the four-point condition gives $d(z,x) = k + 1$. By symmetry, $d(z, y) = k + 1$. Thus z lies on a shortest path joining x and y. Obviously, $track(x)$ and $track(y)$ are independent edges due to $d(x, y) = 2k + 2$ and $d(x,z) = d(y,z) = k + 1$.

Now let s_1s_2 and t_1t_2 be independent edges in F. Let $z - s_1 - s_2 - \cdots - w_1$ and $z - t_1 - t_2 - \cdots - w_2$ be shortest paths of length at least $k + 1$. We will prove $d(w_1, w_2) = 2k + 2$. Since $s_2 - s_1 - z - t_1 - t_2$ is induced we get $d(s_2, t_2) = 4$. Using Lemma 4.3 we obtain the following distance sums:

$$d(w_1, z) + d(s_2, t_2) = 4 + d(w_1, z) \in \{k + 5, k + 6\},$$

$$d(w_1, s_2) + d(z, t_2) = 2 + d(w_1, s_2) \in \{k + 1, k + 2\},$$

$$d(w_1, t_2) + d(z, s_2) = 2 + d(w_1, t_2).$$

Since the difference between the first and second distance sum is at least three the four-point condition implies that the larger two sums must be equal, i.e. the first and third one. So we get

$$k + 3 \leqslant d(w_1, t_2) \leqslant k + 4 \quad \text{and} \quad k + 3 \leqslant d(w_2, s_2) \leqslant k + 4$$

by symmetry. Together with $d(s_2, t_2) = 4$ this implies

$$d(w_1, w_2) + d(s_2, t_2) = 4 + d(w_1, w_2),$$

$$d(w_1, s_2) + d(w_2, t_2) \in \{2k - 2, 2k - 1, 2k\},$$

$$d(w_1, t_2) + d(w_2, s_2) \in \{2k + 6, 2k + 7, 2k + 8\}.$$

By the same argument as above the four-point condition implies that the first and the third distance sum must be equal, i.e. $d(w_1, w_2) \geqslant 2k + 2$. □

Therefore, the following algorithm correctly computes the diameter and a diametral pair of a distance-hereditary graph:

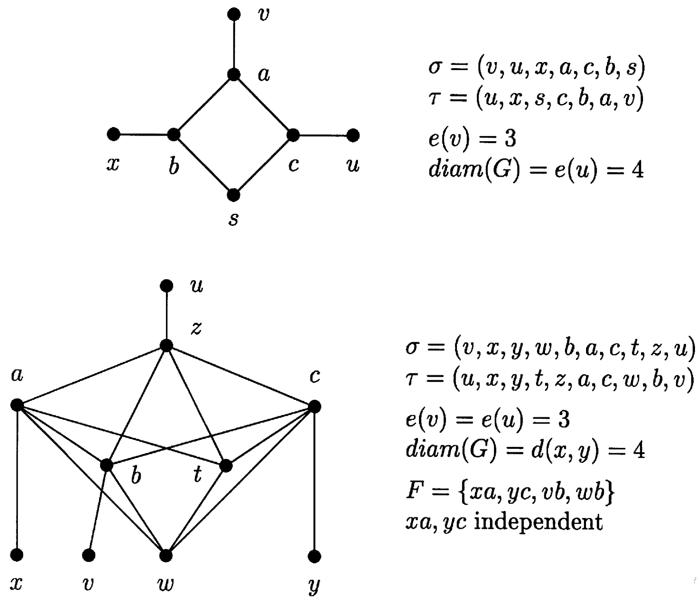


Fig. 3. Algorithm DHGDiam — Examples.

Algorithm DHGDiam.

Input: A connected distance-hereditary graph G.
Output: $diam(G)$ and a diametral pair of vertices of G.

(1) **begin** $\sigma :=$ LexBFS(G,s) for some $s \in V(G)$.
(2) Let v be the first vertex of σ.
(3) **if** $e(v)$ is even **then return**$(e(v),(v,w))$ where $w \in N^{e(v)}(v)$.
(4) **else** $\tau :=$ LexBFS(G,v).
(5) Let u be the first vertex of τ.
(6) **if** $e(u) = e(v)+1$ **then return** $(e(u),(u,w))$ where $w \in N^{e(u)}(u)$.
(7) **else** Let $k \in \mathbb{N}$ such that $e(v) = e(u) = 2k+1$.
(8) Choose a vertex z from $D(u,k) \cap D(v,k+1)$.
(9) $F := \{track(w)\colon w \in V \backslash D(z,k)\}$.
(10) **if** F contains a pair e_1, e_2 of independent edges
(11) **then return**$(2k+2,(x,y))$
where $x,y \in V$ such that $track(x) = e_1$ and $track(y) = e_2$.
(12) **else return**$(2k+1,(v,u))$
(13) **end.**

Before going into the implementation details consider the examples of Fig. 3. In the first one, a $C_4^{(1)}$ minus a pendant vertex, the algorithm correctly stops in step (6). In the second one both first vertices of both LexBFS-ordering s have odd eccentricity. Thus we must compute the *track*-values and the set F.

It remains to show that the above algorithm can be implemented in linear time. It is well known that LexBFS and BFS run in linear time. So it is sufficient to consider steps (9) and (10).

Step (9): At first we build a BFS-tree rooted at z yielding the set of neighbourhoods $N^i(z)$, $i=0,\dots,e(z)$ of z. For any vertex $x \in V\backslash\{z\}$ let $f(x)$ denote the father of x in the BFS-tree. We compute the *track*-values levelwise: For all vertices w in $N^2(z)$ define $track(w):=wy$ where $y=f(w)$. Recursively, we compute $track(w):=track(f(w))$ for $w \in N^i(z)$, $i=3,\dots,e(z)$. Now we can compute F by collecting all *track*-edges of the vertices of the set $V\backslash D(z,k)$. Obviously, the above procedure runs in linear time.

Step (10): We use the BFS-tree rooted at z which was already computed in step (9). Let $b : V \to \mathbb{N}$ be the numbering of the vertices of G produced by BFS where $b(z) = 1$. Let $S_1(S_2)$ be the vertices of $N(z)$ ($N^2(z)$) which are endpoints of edges of F. Furthermore, for all vertices y of S_1 let $C(y)$ be the set of children of y in the BFS-tree contained in S_2, i.e. $C(y)=\{w \in S_2\colon\ y=f(w)\}$. Define $c(y):=|C(y)|$. Obviously, the sets S_1, S_2 and the values $c(y)$ for $y \in S_1$ can be computed in linear time.

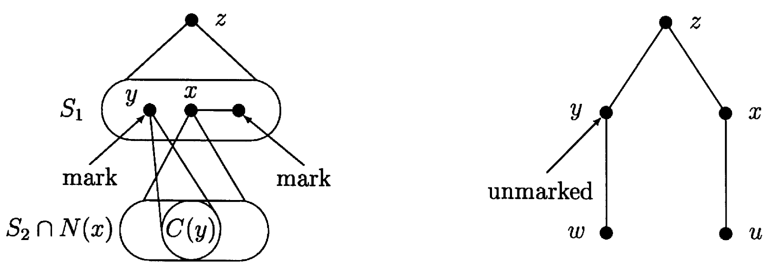
In what follows we explain a procedure looking for a pair of independent edges:

Consider the vertex x of S_1 with maximal b-number. Stepping through the neighbourhood of x we mark all vertices y of S_1 which are either neighbours of x or every BFS-child of y in S_2 is adjacent to x, i.e. $C(y)\subseteq N(x)$.

We show that this can be done in $O(deg(x))$ by using a counter $m(y)$ for each vertex y of S_1 counting the neighbours of x in $C(y)$. Initially, $m(y)=0$ for all $y \in S_1$.

The algorithm steps three times through the neighbourhood of x and thus runs in $O(deg(x))$ time:

- For every neighbour w of x do:
  If $w \in S_1$ then mark w.
  If $w \in S_2$ then increase the counter m of $f(w)$ by 1.
- For every neighbour w of x do:
  If $w \in S_2$ and $c(f(w))=m(f(w))$ then mark $f(w)$.
- For every neighbour w of x do:
  If $w \in S_2$ then reset the counter m of $f(w)$ to zero.



If there is an unmarked vertex $y \in S_1$ then $xy \notin E$ and there must be a BFS-child w of y in S_2 not adjacent to x. We claim that the edges yw and xu, for some neighbour u of x in S_2, are independent (note that $x \in S_1$ implies that there is some vertex $u \in S_2$

with $x = f(u)$ by the definitions of S_1 and S_2). Indeed, since $b(x) > b(y)$ and $x = f(u)$ the rules of BFS imply $uy \notin E$ (if $uy \in E$ then $f(u) = y$). Now $uw \notin E$ for otherwise the set $\{z, x, y, w, u\}$ induces a cycle of length five. Therefore, the edges yw and xu are independent.

Now assume that all vertices of S_1 are marked. Then, by the rules of the marking algorithm, x cannot be an endpoint of a pair of independent edges. So we delete x from S_1 and all neighbours of x in S_2. We repeat the above procedure until we get a pair of independent edges or S_1 is empty.

Since the processing of a vertex x of S_1 takes $O(deg(x))$ the total running time of step (10) is linear.

Summarizing the above we get

Theorem 4.5. *For distance-hereditary graph s the diameter and a diametral pair of vertices can be computed in linear time.*

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