Discrete Mathematics 342 (2019) 326–338

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Obstructions to a small hyperbolicity in Helly graphs

Feodor F. Dragan *, Heather M. Guarnera

Department of Computer Science, Kent State University, Kent, OH, USA

A R T I C L E I N F O

Article history:
Received 12 September 2017
Received in revised form 28 September 2018
Accepted 9 October 2018
Available online 5 November 2018

Keywords:
Hyperbolicity
Structural properties
Structural obstructions
Injective hull
Helly graphs
Isometric subgraphs
King-grid

A B S T R A C T

The δ-hyperbolicity of a graph is defined by a simple 4-point condition: for any four vertices u, v, w, and x, the two larger of the distance sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(v, w)$ differ by at most $2\delta \geq 0$. Hyperbolicity can be viewed as a measure of how close a graph is to a tree metrically; the smaller the hyperbolicity of a graph, the closer it is metrically to a tree. A graph G is Helly if its disks satisfy the Helly property, i.e., every family of pairwise intersecting disks in G has a common intersection. It is known that for every graph G there exists the smallest Helly graph $\mathcal{H}(G)$ into which G isometrically embeds ($\mathcal{H}(G)$ is called the injective hull of G) and the hyperbolicity of $\mathcal{H}(G)$ is equal to the hyperbolicity of G. Motivated by this, we investigate structural properties of Helly graphs that govern their hyperbolicity and identify three isometric subgraphs of the King-grid as structural obstructions to a small hyperbolicity in Helly graphs.

1. Introduction

The δ-hyperbolicity of a graph can be viewed as a measure of how close a graph is to a tree metrically; the smaller the hyperbolicity of a graph, the closer it is metrically to a tree. Recent empirical studies indicated that a large number of real-world networks, including Internet application networks, web networks, collaboration networks, social networks and biological networks, have small hyperbolicity [1,2,10,29,31,35,36,38]. This motivates much research to understand the structure and characteristics of hyperbolic graphs [1,3,6–8,11,13,16,32,39,41], as well as algorithmic implications of small hyperbolicity [8,11–14,18,23,25,33,39]. One aims at developing approximation algorithms for certain optimization problems whose approximation factor depends only on the hyperbolicity of the input graph. To the date such approximation algorithms exist for radius and diameter [11], minimum ball covering [14], p-centers [23], sparse additive spanners [12], the Traveling Salesmen Problem [33], to name a few, which all have an approximation ratio that depends only on the hyperbolicity of the input graph. Notably, there is a quasilinear time algorithm [23] for the p-center problem with additive error at most 3δ, whereas in general it is known [27] that determining an α-approximate solution to p-centers is NP-hard whenever $\alpha < 2$. In another example, there is a linear time algorithm [11] that for any graph G finds a vertex v with eccentricity at most $rad(G) + 5\delta$ (almost the radius of G) and a pair of vertices u, v such that the distance between u and v is at least $diam(G) - 2\delta$ (almost the diameter of G), where δ is the hyperbolicity parameter of G.

In this paper, we are interested in understanding what structural properties of graphs govern their hyperbolicity and in identifying structural obstructions to a small hyperbolicity. It is a well-known fact that the treewidth of a graph G is always greater than or equal to the size of the largest square grid minor of G. Furthermore, in the other direction, the celebrated grid minor theorem by Robertson and Seymour [37] says that there exists a function f such that the treewidth is at most $f(r)$ where r is the size of the largest square grid minor. To the date the best bound on $f(r)$ is $O(r^{98+o(1)})$: every graph of treewidth

* Corresponding author.
E-mail addresses: dragan@cs.kent.edu (F.F. Dragan), hmichaud@kent.edu (H.M. Guarnera).

https://doi.org/10.1016/j.disc.2018.10.017
0012-365X/

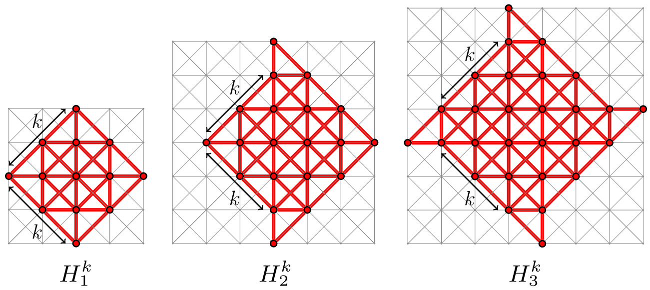

Fig. 1. Graphs H_1^k, H_2^k, and H_3^k in red, where $k = 2$. Each graph is shown isometrically embedded into the King-grid, which is a strong product of two paths and is a particular Helly graph . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

larger than $f(r)$ contains an $(r \times r)$ grid as a minor [9]. Can similar "obstruction" results be proven for the hyperbolicity parameter?

We show in this paper that the *thinness of metric intervals* governs the hyperbolicity of a Helly graph and that *three isometric subgraphs of the King-grid* are the only obstructions to a small hyperbolicity in Helly graphs. Our interest in Helly graphs (the graphs in which disks satisfy the Helly property) stems from the following two facts. We formulate them in the context of graphs although they are true for any metric space. For every graph G there exists the smallest Helly graph $\mathcal{H}(G)$ into which G isometrically embeds; $\mathcal{H}(G)$ is called the *injective hull* of G [22,28]. If G is a δ-hyperbolic graph, then $\mathcal{H}(G)$ is also δ-hyperbolic and every vertex of $\mathcal{H}(G)$ is within distance 2δ from some vertex of G [34]. Thus, from our main result for Helly graphs (see Theorem 3), one can state the following.

Theorem 1. *An arbitrary graph G has hyperbolicity at most δ if and only if its injective hull $\mathcal{H}(G)$ contains*
– no H_2^{δ}, when δ is an integer,
– neither $H_1^{\delta+\frac{1}{2}}$ nor $H_3^{\delta-\frac{1}{2}}$, when δ is a half-integer,
from Fig. 1 as an isometric subgraph.

The injective hull here can be viewed as playing a similar role as the minors in the grid minor theorem for treewidth by Robertson and Seymour. Helly graphs play a similar role for the hyperbolicity as chordal graphs play for the treewidth. Note that each of the graphs H_1^k, H_2^k, H_3^k contains a square grid of side k (see Fig. 1) as an isometric subgraph. Thus, if the hyperbolicity of a Helly graph is large then it has a large square grid as an isometric subgraph. This result (along with a connection between the treewidth and the hyperbolicity established in [17][1]) calls for an attempt to develop a theory similar to the bidimensionality theory (see survey [19] and papers cited therein). The bidimensionality theory builds on the graph minor theory of Robertson and Seymour by extending the mathematical results and building new algorithmic tools. Using algorithms for graphs of bounded treewidth as sub-routines (see also an earlier paper [24]), it provides general techniques for designing efficient fixed-parameter algorithms and approximation algorithms for NP-hard graph problems in broad classes of graphs. This theory applies to graph problems that are "bidimensional" in the sense that (1) the solution value for the $k \times k$ grid graph (and similar graphs) grows with k, typically as $\Omega(k^2)$, and (2) the solution value goes down when contracting edges, and optionally when deleting edges, in the graph. Examples of such problems include feedback vertex set, vertex cover, minimum maximal matching, face cover, a series of vertex-removal parameters, (edge) dominating set, connected (edge) dominating set, unweighted TSP tour, and chordal completion (fill-in). We are currently investigating this direction.

Previously, it was known that the hyperbolicity of median graphs is controlled by the size of isometrically embedded square grids (see [4,7]), and recently [7] showed that the hyperbolicity of weakly modular graphs (a far reaching superclass of the Helly graphs) is controlled by the sizes of metric triangles and isometric square grids: if G is a weakly modular graph in which any metric triangle is of side at most μ and any isometric square grid contained in G is of side at most ν, then G is $O(\nu+\mu)$-hyperbolic. Recall that three vertices x, y, z of a graph form a metric triangle if for each vertex $v \in \{x, y, z\}$, any two shortest paths connecting it with the two other vertices from $\{w, y, z\}$ have only v in common. Projecting this general result to Helly graphs (where $\mu \le 1$) one gets only that every Helly graph with isometric grids of side at most ν is $c\nu$-hyperbolic with a constant c larger than 1 (about 8).

Injective hulls of graphs were recently used in [13] to prove a conjecture by Jonckheere et al. [30] that real-world networks with small hyperbolicity have a core congestion. It was shown [13] that any finite subset X of vertices in a locally finite δ-hyperbolic graph G admits a disk $D(m, 4\delta)$ centered at vertex m, which intercepts all shortest paths between at least one half of all pairs of vertices of X.

1 In fact, [17] establishes a relation between the treelength and the treewidth of a graph but according to [11] the hyperbolicity and the treelength are within a factor of $O(\log n)$ from each other.

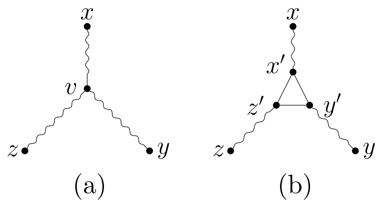

Fig. 2. Vertices x, y, z and three shortest paths connecting them in pseudo-modular graphs.

There has also been much related work on the characterization of δ-hyperbolic graphs via forbidden isometric subgraphs — particularly, when $\delta = \frac{1}{2}$. Koolen and Moulton [32] provide such a characterization for $\frac{1}{2}$-hyperbolic bridged graphs via six forbidden isometric subgraphs. Bandelt and Chepoi [3] generalize these results to all $\frac{1}{2}$-hyperbolic graphs via the same forbidden isometric subgraphs and the property that all disks of G are convex. Additionally, Coudert and Ducoffe [16] prove that a graph is $\frac{1}{2}$-hyperbolic if and only if every graph power G^i is C_4-free for $i \geq 1$, and one additional graph is C_4-free. Brinkmann et al. [6] characterize $\frac{1}{2}$-hyperbolic chordal graphs via two forbidden isometric subgraphs. Wu and Zhang [41] prove that a 5-chordal graph is $\frac{1}{2}$-hyperbolic if and only if it does not contain six isometric subgraphs. Cohen et al. [15] prove that a biconnected outerplanar graph is $\frac{1}{2}$-hyperbolic if and only if either it is isomorphic to C_5 or it is chordal and does not contain a forbidden subgraph. We present a characterization of δ-hyperbolic Helly graphs, for every δ, with three forbidden isometric subgraphs. Further characterizations of Helly graphs with small hyperbolicity constant are deduced from our main result.

2. Preliminaries

We use the terminology and definitions as described in standard graph theory textbooks [20,40]. All graphs $G = (V, E)$ appearing here are connected, finite, unweighted, undirected, loopless and without multiple edges. The *length of a path* from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest path connecting u and v in G. We omit the subscript when G is known by context. For a subset $A \subseteq V$, a subgraph $G(A)$ of a graph G *induced* by A is defined as $G(A) = (A, E')$ where $uv \in E'$ if and only if $u, v \in A$ and $uv \in E$. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as their distance in G. The *kth power* G^k of G is defined as $G^k = (V, E')$ where $E' = \{uv : u, v \in V \text{ and } d(u, v) \leq k\}$. A *disk* $D(v, r)$ of a graph G centered at a vertex $v \in V$ and with radius r is the set of all vertices with distance no more than r from v (i.e., $D(v, r) = \{u \in V : d_G(v, u) \leq r\}$). For any two vertices u, v of G, $I(u, v) = \{z \in V : d(u, v) = d(u, z) + d(z, v)\}$ is the (metric) *interval* between u and v, i.e., all vertices that lay on shortest paths between u and v.

A family $\mathcal{F}$ of sets S_i has the *Helly property* if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$ the following holds: if the elements of $\mathcal{F}'$ pairwise intersect, then the intersection of all elements of $\mathcal{F}'$ is also non-empty. A graph is called *Helly* if its family of all disks $\mathcal{D}(G) = \{D(v, r) : v \in V, r \in N\}$ satisfies the Helly property. Note that two disks $D(v, p)$ and $D(u, q)$ intersect each other if and only if $d_G(u, v) \leq p + q$. Two disks $D(v, p)$ and $D(u, q)$ of G are said to *see* each other, sometimes also referred to as *touching* each other, if they intersect or there is an edge in G with one end in $D(v, p)$ and other end in $D(u, q)$ (equivalently, if $d_G(u, v) \leq p + q + 1$). The *strong product* of a set of graphs G_i for $i = 1, 2, \ldots, k$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the vertex sets V_i, and there is an edge between vertices $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$ if and only if a_i is either equal or adjacent to b_i for $i = 1, 2, \ldots, k$. A *King-grid* is a strong product of two paths. King-grids form a natural subclass of Helly graphs.

The following lemma will be frequently used in this paper. It is true for a larger family of pseudo-modular graphs but we will use it in the context of Helly graphs. Pseudo-modular graphs are exactly the graphs where each family of three pairwise intersecting disks has a common intersection [5]. Clearly, Helly graphs is a subclass of pseudo-modular graphs.

Lemma 1 ([5]). *For every three vertices x, y, z of a pseudo-modular graph G there exist three shortest paths $P(x, y)$, $P(x, z)$, $P(y, z)$ connecting them such that either (1) there is a common vertex v in $P(z, y) \cap P(x, z) \cap P(x, y)$ or (2) there is a triangle $\triangle(x', y', z')$ in G with edge $z'y'$ on $P(z, y)$, edge $x'z'$ on $P(x, z)$ and edge $x'y'$ on $P(x, y)$ (see Fig. 2). Furthermore, (1) is true if and only if $d(x, y) = p + q$, $d(x, z) = p + k$ and $d(y, z) = q + k$, for some $k, p, q \in N$, and (2) is true if and only if $d(x, y) = p + q + 1$, $d(x, z) = p + k + 1$ and $d(y, z) = q + k + 1$, for some $k, p, q \in N$.*

We are interested in hyperbolic graphs (sometimes referred to as graphs with a negative curvature). δ-Hyperbolic metric spaces have been defined by Gromov [26] in 1987 via a simple 4-point condition: for any four points u, v, w, x, the two larger of the distance sums $d(u, v)+d(w, x)$, $d(u, w)+d(v, x)$, $d(u, x)+d(v, w)$ differ by at most $2\delta \geq 0$. They play an important role in geometric group theory and in the geometry of negatively curved spaces, and have recently become of interest in several domains of computer science, including algorithms and networking. A connected graph $G = (V, E)$ equipped with standard graph metric d_G is δ-*hyperbolic* if the metric space (V, d_G) is δ-hyperbolic. The smallest value δ for which G is δ-hyperbolic is called the *hyperbolicity* of G and denoted by $hb(G)$. Let also $hb(u, v, w, x)$ $(u, v, w, x \in V)$ denote one half of the difference

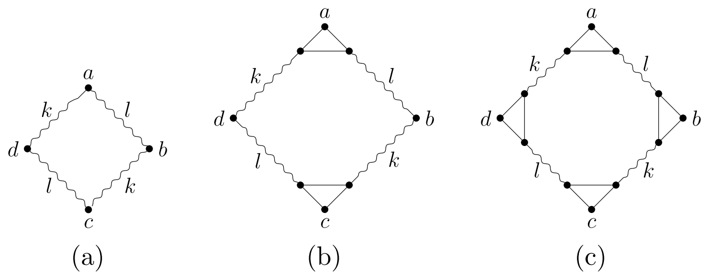

Fig. 3. $\{a, b, c, d\}$-distance preserving subgraphs.

between the two larger distance sums from $d(u, v)+d(w, x)$, $d(u, w)+d(v, x)$, $d(u, x)+d(v, w)$. The Gromov product of two vertices $x, y \in V$ with respect to a third vertex $z \in V$ is defined as $(x|y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$. The focus of this paper is primarily on *δ-hyperbolic Helly graphs* (i.e., graphs which satisfy the Helly property as well as have δ hyperbolicity).

Using the Gromov product, we can reformulate Lemma 1. Since $d(x, y) = (x|z)_y + (z|y)_x$, it is easy to check that for any three vertices x, y, z of an arbitrary graph, either all products $(y|z)_x, (y|x)_z, (x|z)_y$ are integers or all are half-integers.

Lemma 2. *For every three vertices x, y, z of a pseudo-modular graph G there exist three shortest paths $P(z, y)$, $P(x, z)$, $P(x, y)$ connecting them such that either (1) there is a common vertex v in $P(z, y) \cap P(x, z) \cap P(x, y)$ or (2) there is a triangle $\triangle(x', y', z')$ in G with edge $z'y'$ on $P(z, y)$, edge $x'z'$ on $P(x, z)$ and edge $x'y'$ on $P(x, y)$ (see Fig. 2). Furthermore, (1) is true if and only if $(x|y)_z$ is an integer and $(x|y)_z = d(z, v)$, and (2) is true if and only if $(x|y)_z$ is a half-integer and $\lfloor (x|y)_z \rfloor = d(z, z')$.*

Proof. Let $\alpha_z = (x|y)_z$, $\alpha_x = (z|y)_x$, and $\alpha_y = (z|x)_y$. We have $d(z, x) - \alpha_z = d(z, x) - \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)) = \frac{1}{2}(d(x, z) + d(x, y) - d(y, z)) = (z|y)_x$. Similarly, $(z|x)_y = d(z, y) - \alpha_z$. Therefore, α_x and α_y are integers if and only if α_z is an integer.

Let there be a common vertex v in $P(z, y) \cap P(x, z) \cap P(x, y)$. By definition, $2\alpha_z = (d(x, v) + d(v, z)) + (d(z, v) + d(v, y)) - (d(x, v) + d(v, y)) = 2d(v, z)$. Thus, $d(v, z) = \alpha_z$, and since $d(v, z)$ is an integer then α_z is an integer. The converse follows from Lemma 1 for $p = \alpha_x$, $q = \alpha_y$ and $k = \alpha_z$.

Let there be a triangle $\triangle(x', y', z')$ in G with edge $z'y'$ on $P(z, y)$, edge $x'z'$ on $P(x, z)$ and edge $x'y'$ on $P(x, y)$. By definition, $2\alpha_z = (d(x, x')+1+d(z', z))+(d(z, z')+1+d(y', y))-(d(x, x')+1+d(y', y)) = 2d(z, z')+1$. Thus, $d(z, z') = \lfloor \alpha_z \rfloor$, and since $2d(z, z') + 1$ is odd then α_z is a half-integer. The converse follows from Lemma 1 for $p = \lfloor \alpha_x \rfloor$, $q = \lfloor \alpha_y \rfloor$ and $k = \lfloor \alpha_z \rfloor$. □

The set $S_k(x, y) = \{z \in I(x, y) : d(z, x) = k\}$ is called a *slice* of the interval from x to y. The diameter of a slice $S_k(x, y)$ is the maximum distance in G between any two vertices of $S_k(x, y)$. An interval $I(x, y)$ is said to be τ-thin if diameters of all slices $S_k(x, y)$, $k \in N$, of it are at most τ. A graph G is said to have τ-thin intervals if all intervals of G are τ-thin. The smallest τ for which all intervals of G are τ-thin is called the *interval thinness* of G and denoted by $\tau(G)$. That is,

$$\tau(G) = \max\{d(u, v) : u, v \in S_k(x, y), x, y \in V, k \in N\}.$$

The following lemma is a folklore and easy to show using the definition of hyperbolicity.

Lemma 3. *For any graph G, $\tau(G) \leq 2hb(G)$.*

Proof. Consider any interval $I(x, y)$ in G and arbitrary two vertices $u, v \in S_k(x, y)$. Consider the three distance sums $S_1 = d(x, y) + d(u, v)$, $S_2 = d(x, u) + d(y, v)$, $S_3 = d(x, v) + d(y, u)$. As $u, v \in S_k(x, y)$, we have $S_2 = S_3 = d(x, y) \leq S_1$. Hence, $2hb(G) \geq S_1 - S_2 = d(x, y) + d(u, v) - d(x, y) = d(u, v)$ for any two vertices from the same slice of G, i.e., $2hb(G) \geq \tau(G)$. □

3. Thinness of intervals governs the hyperbolicity of a Helly graph

A qualitative relationship between hyperbolicity and thinness of intervals is easy to show even for a more general class of median graphs. The true contribution of our paper is more quantitative. In fact, we obtain the exact relationship between the two. We focus now on demonstrating that the converse of Lemma 3 for Helly graphs is also true such that the value of $2hb(G)$ is upper bounded by $\tau(G) + 1$. Note that, for general graphs G, the values of $\tau(G)$ and $2hb(G)$ can be very far from each other. Consider an odd cycle with $4k + 1$ vertices; each pair of vertices has a unique shortest path, so no two vertices are in the same slice. Thus $\tau(G) = 0$ and $2hb(G) = 2k$.

We say that a graph $G' = (V', E')$ with $\{a, b, c, d\} \subset V'$ is an *$\{a, b, c, d\}$-distance-preserving subgraph* of a graph G if $d_G(x, y) = d_{G'}(x, y)$ for every pair of vertices x, y from $\{a, b, c, d\}$. In Fig. 3(c), an $\{a, b, c, d\}$-distance-preserving subgraph of a graph G with $d_G(a, c) = d_G(b, d) = k + l + 3$, $d_G(a, b) = d_G(c, d) = l + 2$, and $d_G(b, c) = d_G(d, a) = k + 2$ is shown.

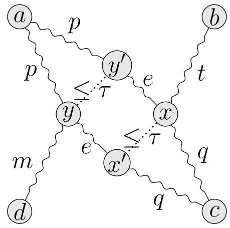

Fig. 4. Illustration for Case 1 for the proof of Lemma 4.

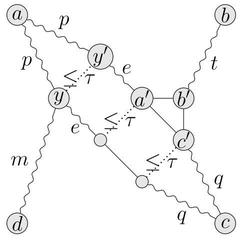

Fig. 5. Illustration for Case 2 for the proof of Lemma 4.

Lemma 4. *For every Helly graph G, $2hb(G) \leq \tau(G)+1$. Furthermore, for any Helly graph G, $2hb(G) = \tau(G)+1$ if and only if $\tau(G)$ is odd and there exists in G an $\{a, b, c, d\}$-distance-preserving subgraph for some $\{a, b, c, d\}$ as depicted in Fig. 3(c) with $k = l = \lfloor \frac{\tau(G)}{2} \rfloor$.*

Proof. Consider arbitrary four vertices a, b, c, d with $hb(G) = hb(a, b, c, d) =: \delta$ and let $d_G(a, c) + d_G(b, d) \geq d_G(a, b) + d_G(c, d) \geq d_G(a, d) + d_G(b, c)$. Let also $\tau := \tau(G)$. We apply Lemma 1 once to vertices $\{a, b, c\}$ and again to vertices $\{a, d, c\}$. By Lemma 1, a set of three vertices with some shortest paths connecting them define either configuration (1) or configuration (2) from Fig. 2. Hence, there are three cases, up-to symmetry, to consider.

Case 1. *For vertices a, b, c there are three shortest paths $P(a, b)$, $P(b, c)$, $P_b(a, c)$ that share a common vertex x. For vertices a, d, c there are three shortest paths $P(a, d)$, $P(d, c)$, $P_d(a, c)$ that share a common vertex y. Note that we use the notation $P_b(a, c)$ and $P_d(a, c)$ here to distinguish between the two shortest (a, c)-paths that exist by applying Lemma 1 to vertices a, b, c and again separately to vertices a, d, c, respectively.*

This situation is shown in Fig. 4. It is unknown if x and y are on the same slice of $I(a, c)$ or not, so we consider vertices $y' \in P_b(a, c)$ and $x' \in P_d(a, c)$ with $d_G(a, y) = d_G(a, y') =: p$ and $d_G(c, x) = d_G(c, x') =: q$. Set also $e := d_G(y', x) = d_G(y, x')$, $m := d_G(d, y)$, $t := d_G(b, x)$ (see Fig. 4). Vertices x, x' lie on the same slice of $I(a, c)$, as do y, y'. Given that intervals of G are τ-thin, we get $2\delta = d_G(a, c) + d_G(b, d) - (d_G(a, b) + d_G(c, d)) \leq p + e + q + t + \tau + e + m - (p + e + t + m + e + q) = \tau$, i.e., $2\delta \leq \tau$.

Case 2. *For vertices a, b, c there are three shortest paths $P(a, b)$, $P(b, c)$, $P_b(a, c)$ and a triangle $\triangle(b', c', a')$ in G with edge $a'b'$ on $P(a, b)$, edge $b'c'$ on $P(b, c)$ and edge $a'c'$ on $P_b(a, c)$. For vertices a, d, c there are three shortest paths $P(a, d)$, $P(d, c)$, $P_b(a, c)$ that share a common vertex y.*

This situation is shown in Fig. 5. Consider vertex $y' \in P_b(a, c)$ such that $p := d_G(a, y') = d_G(a, y)$, and let now $q := d_G(c, c')$, set $e := d_G(y', a')$, and $t := d_G(b, b')$. Since intervals of G are τ-thin, we get $2\delta = d_G(a, c) + d_G(b, d) - (d_G(a, b) + d_G(c, d)) \leq p + e + 1 + q + t + 1 + \tau + e + m - (p + e + 1 + t + m + e + 1 + q) = \tau$, i.e., $2\delta \leq \tau$.

Case 3. *For vertices a, b, c there are three shortest paths $P(a, b)$, $P(b, c)$, $P_b(a, c)$ and a triangle $\triangle(b', c', a')$ in G with edge $a'b'$ on $P(a, b)$, edge $b'c'$ on $P(b, c)$ and edge $a'c'$ on $P_b(a, c)$. For vertices a, d, c there are three shortest paths $P(a, d)$, $P(d, c)$, $P_d(a, c)$ and a triangle $\triangle(d', c'', a'')$ in G with edge $a''d'$ on $P(a, d)$, edge $d'c''$ on $P(d, c)$ and edge $a''c''$ on $P_d(a, c)$.*

This situation is shown in Fig. 6. If vertices a', a'' are not in the same slice of $I(a, c)$ then set $p := d_G(a, a'')$, and let vertex c^* denote the vertex on $P_b(a, c)$ such that $d_G(a, c^*) = p + 1$. Set $e := d_G(c^*, a')$ and $m := d_G(d, d')$. Then, $2\delta = d_G(a, c)+d_G(b, d)-(d_G(a, b)+d_G(c, d)) \leq p+1+e+1+q+t+1+e+\tau+1+m-(p+1+e+1+t+q+1+e+1+m) = \tau$.

If vertices a', a'' are in the same slice of $I(a, c)$ (see Fig. 7 for this special subcase; only in this subcase we may have $2\delta = \tau + 1$) then, using notations from Fig. 7, $2\delta = d_G(a, c) + d_G(b, d) - (d_G(a, b) + d_G(c, d)) \leq p + 1 + q + t + 1 + \tau + 1 + m - (p + 1 + t + m + 1 + q) = \tau + 1$. Furthermore, if $2\delta = \tau + 1$, then $d_G(a', a'') = d_G(c', c'') = \tau$.

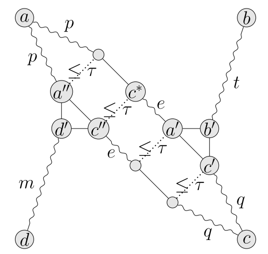

Fig. 6. Illustration for Case 3 for the proof of Lemma 4.

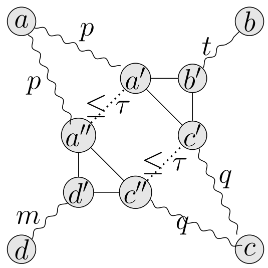

Fig. 7. A special subcase of Case 3.

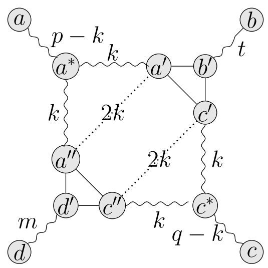

Fig. 8. When $\tau(G) = 2k, \delta(a, b, c, d) = k$.

Assume that $2\delta = \tau + 1$ and τ is even (see Fig. 8). Let $\tau = 2k$. Consider disks $D(a, p - k), D(a'', k), D(a', k)$ in G. These disks pairwise intersect. Hence, there must exist a vertex a^* at distance $p - k$ from a and at distance k from both a' and a''. Similarly, there is a vertex c^* in G at distance $q - k$ from c and at distance k from both c' and c''. These vertices a^* and c^* belong to slice $S_{t+1+k}(b, d)$ of $I(b, d)$. Hence, $d_G(a^*, c^*) \leq \tau = 2k$ must hold. On the other hand, $p + 1 + q = d_G(a, c) \leq d_G(a, a^*) + d_G(a^*, c^*) + d_G(c^*, c) \leq p - k + 2k + q - k = p + q$, a contradiction. Thus, when τ is even, $2\delta = \tau$.

Assume now that $2\delta = \tau + 1$ and τ is odd. Let $\tau = 2k + 1$. As $d_G(a', a'') = 2k + 1$ and $d_G(a, a') = d_G(a, a'') = p$, by Lemma 1, there must exist three shortest paths $P(a, a'), P(a, a''), P(a', a'')$ and a triangle $\triangle(x, y, z)$ in G with edge xy on $P(a, a')$, edge xz on $P(a, a'')$ and edge zy on $P(a', a'')$ (note that $P(a, a'), P(a, a''), P(a', a'')$ cannot have a common vertex because of distance requirements). Similarly, there must exist three shortest paths $P(c, c'), P(c, c''), P(c', c'')$ and a triangle $\triangle(u, v, w)$ in G with edge uv on $P(c, c')$, edge uw on $P(c, c'')$ and edge vw on $P(c', c'')$. Thus, by distance requirements, four triangles $\triangle(x, y, z), \triangle(a', b', c'), \triangle(u, v, w), \triangle(d', a'', c'')$ with corresponding shortest paths $P(y, a') \subseteq P(a, a'), P(a'', z) \subseteq P(a'', a)$, $P(c'', w) \subseteq P(c'', c), P(c', v) \subseteq P(c', c)$ of length $k = \lfloor \frac{\tau(G)}{2} \rfloor$ each form in G an $\{x, b', u, d'\}$-distance-preserving subgraph isomorphic to the one depicted in Fig. 3(c) with $k = l$.

To complete the proof, it is enough to verify that if $\tau(G)$ is odd and there exists in G an $\{a, b, c, d\}$-distance-preserving subgraph depicted in Fig. 3(c) with $k = l = \lfloor \frac{\tau(G)}{2} \rfloor$, then we obtain $2hb(a, b, c, d) = \tau(G) + 1$. □

The following lemmas prove that the three $\{a, b, c, d\}$-distance preserving subgraphs shown in Fig. 3 can be isometrically embedded into three Helly graphs termed $H_1^{k,l}, H_2^{k,l}$, and $H_3^{k,l}$, respectively. Each of $H_1^{k,l}, H_2^{k,l}$, and $H_3^{k,l}$ is an isometric subgraph of a King-grid (Fig. 9 gives small examples for $k = l = 2$). Each is induced by the vertices in red as demonstrated in Fig. 9(a), Fig. 9(b), and Fig. 9(c) such that its four extreme vertices correspond to the four extreme vertices of an $\{a, b, c, d\}$-distance preserving subgraph shown in Fig. 3(a), Fig. 3(b), and Fig. 3(c), respectively. In the description that follows let vertices of the form x_y and x_z denote neighbors which are adjacent to vertex x such that $x_y \in I(x, y)$ and $x_z \in I(x, z)$. Thus, in $H := H_1^{k,l}$ we

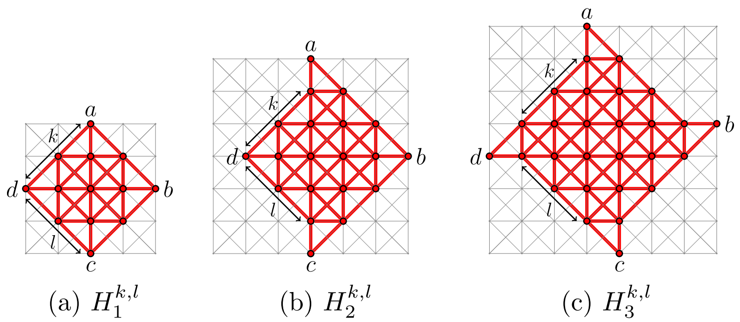

Fig. 9. Examples of H_1^k, H_2^k, and H_3^k shown in red, where $k = l = 2$, based on respective inputs from Fig. 3. We omit the second superscript and use the notation H_i^k when $k = l$. Isometric embeddings of those graphs into the King-grid are shown . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

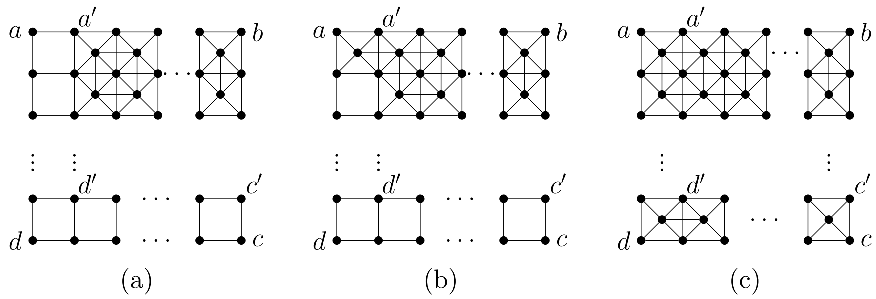

Fig. 10. Extension of an isometric subgraph $H_1^{k-1,l-1}$ to an isometric subgraph $H_1^{k,l}$.

have that $d_H(a, d) = d_H(b, c) = k$ and $d_H(a, b) = d_H(d, c) = l$. In $H := H_2^{k,l}$, we have that $d_H(a_d, d) = d_H(b, c_b) = k$ and $d_H(a_b, b) = d_H(d, c_d) = l$. Finally, in $H := H_3^{k,l}$, we have that $d_H(a_d, d_a) = d_H(b_c, c_b) = k$ and $d_H(a_b, b_a) = d_H(d_c, c_d) = l$.

We will show in Section 4 that any Helly graph G with $hb(G) = \delta$ has an isometric H_1^k, H_2^k, or H_3^k, where k is a function of δ. These isometric subgraphs will be equally important as forbidden subgraphs for $hb(G) \leq \delta$ in Section 4. We provide the hellification of all three graphs here for completeness, however, the remainder of this section will use only the graph in Fig. 3(c) and its hellification $H_3^{k,l}$ in order to refine result of Lemma 4 in the special case when $2hb(G) = \tau(G) + 1$.

Lemma 5. *If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted in Fig. 3(a), then G has an isometric subgraph $H_1^{k,l}$ with a, b, c, d as corner points (see Fig. 9(a)).*

Proof. Let a, b, c, d be four vertices of a Helly graph G such that $d_G(a, c) = d_G(b, d) = k + l$ and $d_G(a, b) = d_G(c, d) = l$ and $d_G(b, c) = d_G(d, a) = k$. Let $P(d, a) = (d, v_1, v_2, \dots, v_k = a)$, $P(d, c) = (d, u_1, u_2, \dots, u_l = c)$ be two shortest paths connecting the appropriate vertices. Consider disks $D(v_1, 1)$, $D(u_1, 1)$, $D(b, k + l - 2)$ in G. These disks pairwise intersect. Hence, by the Helly property, there is a vertex d' which is adjacent to both v_1 and u_1 and at distance $k + l - 2$ from b. Since disks $D(a, 1)$, $D(b, l-1)$, $D(d', k-1)$ pairwise intersect, there must exist a vertex a' such that a' is adjacent to a and at distance $k-1$ from d' and distance $l-1$ from b. Similarly, considering pairwise intersecting disks $D(c, 1)$, $D(b, k-1)$, $D(d', l-1)$, there exists a vertex c' which is adjacent to c and at distance $l-1$ from d' and distance $k-1$ from b. For vertices a', b, c', d' we have $d_G(a', c') = d_G(b, d') = l + k - 2$ and $d_G(a', b) = d_G(c', d') = l - 1$ and $d_G(b, c') = d_G(d', a') = k - 1$. Hence, by induction, we may assume that in G there is an isometric subgraph $H_1^{k-1,l-1}$ with a', b, c', d' as corner points. In what follows, using the Helly property, we extend this $H_1^{k-1,l-1}$ to isometric $H_1^{k,l}$ with a, b, c, d as corner points (see Fig. 10 for an illustration).

Let $P(d', a') = (d' = v'_1, v'_2, \dots, v'_k = a')$ be the shortest path of $H_1^{k-1,l-1}$ connecting d' with a'. For each edge $v'_i v'_{i+1}$ of this path, denote by w_i a vertex of $H_1^{k-1,l-1}$ which forms a triangle with $v'_i v'_{i+1}$. Let $P(d, a)$ denote path $(d = v_0, v_1, v_2, \dots, v_k = a)$. First, we show that each vertex $v_i \in P(d, a)$ for $i = 1, 2, \dots, k-1$ can be chosen such that $v_i v'_i$ is an edge of G for each i. Let $i \geq 1$ be the smallest index such that $v_i v'_i \notin E$. Consider pairwise intersecting disks $D(v_{i-1}, 1)$, $D(v'_i, 1)$, $D(a, d_G(a, v_i))$. By the Helly property, there is a vertex v_i^* in G which is adjacent to both v_{i-1} and v'_i and at distance $d_G(a, v_i)$ from a. Hence, we can replace part of $P(d, a)$ from v_i to a with a new shortest path from v_i^* to a. So, we can assume that $v_i v'_i \in E$ for each i. Since vertices $a, v'_k, w_{k-1}, v'_{k-1}, v_{k-1}$ are pairwise at distance at most 2, by the Helly property, there must exist a vertex w'_{k-1} which is adjacent to all $a, v'_k, w_{k-1}, v'_{k-1}, v_{k-1}$. Having vertex w'_{k-1}, we can use the Helly property to impose a new vertex w'_{k-2} adjacent to all $v_{k-1}, v'_{k-1}, w'_{k-1}, w_{k-2}, v'_{k-2}, v_{k-2}$. Continuing this way, we obtain a new vertex w'_i (for $i = k-3, k-4, \dots, 1$) which is adjacent to all $v_{i+1}, v'_{i+1}, w'_{i+1}, w_i, v'_i, v_i$. This completes the addition to $H_1^{k-1,l-1}$ along the

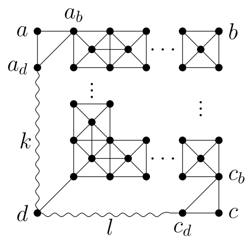

Fig. 11. Using the Helly property, the graph from Fig. 3(b) is shown to have $H_2^{k,l}$ as an isometric subgraph.

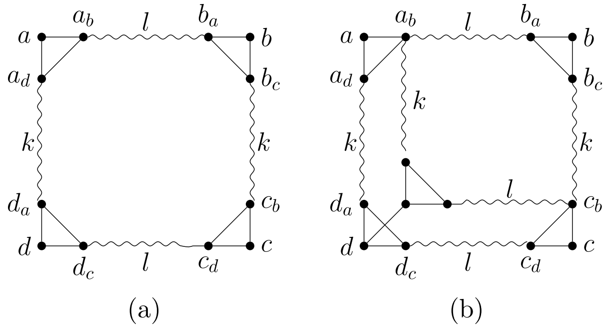

Fig. 12. Using the Helly property, the graph from Fig. 3(c) is shown to have $H_3^{k,l}$ as an isometric subgraph.

path $P(d, a) = (d, v_1, v_2, \ldots, v_k = a)$. Similarly, the addition along the path $P(d, c) = (d, u_1, u_2, \ldots, u_l = c)$ can be done completing the extension of $H_1^{k-1,l-1}$ to $H_1^{k,l}$ which is clearly an isometric subgraph of G. □

Lemma 6. *If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted in Fig. 3(b), then G has an isometric subgraph $H_2^{k,l}$ with a, b, c, d as corner points (see Fig. 9(b)).*

Proof. Let a, b, c, d be vertices of a Helly graph G, with $\triangle(a, a_b, a_d)$ and $\triangle(c, c_b, c_d)$ such that $d_G(a, c) = k+l+2 = d_G(a, a_b)+d_G(a_b, c_b)+d_G(c_b, c) = d_G(a, a_d)+d_G(a_d, c_d)+d_G(c_d, c)$, $d_G(b, d) = k+l+1 = d_G(d, a_d)+1+d_G(a_b, b) = d_G(d, c_d)+1+d_G(c_b, b)$, and $d_G(b, c) = d_G(d, a) = k + 1$ and $d_G(a, b) = d_G(c, d) = l + 1$. Consider disks $D(a_b, k)$, $D(c_b, l)$, and $D(d, 1)$ in G. These disks pairwise intersect. Hence, by the Helly property, there is a vertex d' which is adjacent to d and at distance k from a_b and at distance l from c_b. For vertices a_b, b, c_b, d', we have $d_G(a_b, b) = d_G(c_b, d') = l$, $d_G(b, c_b) = d_G(d', a_b) = k$, and $d_G(a_b, c_b) = d_G(d', b) = k + l$. By Lemma 5, there is an isometric subgraph $H_1^{k,l}$ with a_b, b, c_b, d' as corner points (see Fig. 11).

Let $P(a_b, d') = (a_b = v'_0, v'_1, v'_2, \ldots, v'_k = d')$ be the shortest path of $H_1^{k,l}$ connecting d' with a_b. For each edge $v'_i v'_{i+1}$ of this path, denote by w'_{i+1} a vertex of $H_1^{k,l}$ which forms a triangle with $v'_i v'_{i+1}$. Since vertices d, v'_k, v'_{k-1}, w'_k are pairwise distant at most 2 and distant from a at most $k + 1$, by the Helly property there must exist a vertex v^*_k adjacent to d, v'_k, v'_{k-1}, w'_k and at distance k from a. Having vertex v^*_k, we can use the Helly property to impose a new vertex v^*_{k-1} which is adjacent to all $v^*_k, v'_{k-1}, v'_{k-2}, w'_{k-1}$ and at distance $k - 1$ from a. Continuing this way, we obtain a new vertex v^*_i which is adjacent to all $v^*_{i+1}, v'_i, v'_{i-1}, w'_i$ and at distance i from a (for $i = k - 2, k - 3, \ldots, 1$). This completes the addition to $H_1^{k,l}$ along the path $P(a_b, d')$. Similarly, the addition along the path $P(c_b, d') = (c_b = u'_0, u'_1, u'_2, \ldots, u'_l = d')$ can be done. This completes the extension of $H_1^{k,l}$ to $H_2^{k,l}$.

Clearly, $H_2^{k,l}$ obtained from $H_1^{k,l}$ is an isometric subgraph of G. Recall that $H_2^{k,l}$ is a $\{a, b, c, d\}$-distance preserving subgraph of G. We know from Lemma 5 that $H_1^{k,l}$-part of $H_2^{k,l}$ is an isometric subgraph of G. We know also that every pair $x, y \in H_2^{k,l} \setminus H_1^{k,l}$ belongs to a shortest path of G from a to c passing through d. Finally, every pair x, y with $x \in H_2^{k,l} \setminus H_1^{k,l}$ and $y \in H_1^{k,l}$ belongs to a shortest path of G connecting a with c or b with d. □

Lemma 7. *If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted in Fig. 3(c), then G has an isometric subgraph $H_3^{k,l}$ with a, b, c, d as corner points (see Fig. 9(c)).*

Proof. Let a, b, c, d be vertices of a Helly graph G such that $d_G(a, c) = d_G(b, d) = l+k+3$ and $d_G(a, b) = d_G(c, d) = l+2$ and $d_G(b, c) = d_G(d, a) = k+2$ (see Fig. 12(a)). Since $d(a_b, c_b) = k+l+1$, $d(a_b, d) = k+1+1$ and $d(c_b, d) = 1+l+1$, by Lemma 1, there is a triangle $\triangle(d', d'_a, d'_c)$ such that d is adjacent to d' and $d_G(d'_c, c_b) = l$, $d_G(d'_a, a_b) = k$. For vertices a_b, b, c_b, d', we have $d_G(a_b, b) = d_G(c_b, d') = l + 1$, $d_G(b, c_b) = d_G(d', a_b) = k + 1$, and $d_G(d', b) = k + l + 2$, as well as $d_G(a_b, c_b) = k + l + 1$. By Lemma 6, there is in G an isometric $H_2^{k,l}$ with a_b, b, c_b, d' as corner points (see Fig. 12(b)).

Let $P(a_b, d') = (a_b = v'_0, v'_1, v'_2, \ldots, v'_k, d')$ be the shortest path of $H_2^{k,l}$ connecting d' with a_b, and let $P(c_b, d') = (c_b = u'_0, u'_1, u'_2, \ldots, u'_l, d')$ be the shortest path of $H_2^{k,l}$ connecting d' with c_b. For each edge $v'_i v'_{i+1}$, denote by w_{i+1} a vertex of $H_2^{k,l}$ which forms a triangle with $v'_i v'_{i+1}$. Since vertices d', d, v'_k, u'_l are pairwise at distance at most 2 and at distance at most $k+2$ from a, by the Helly property, there must exist a vertex v^*_{k+1} adjacent to d', d, v'_k, u'_l and at distance $k+1$ from a. Having vertex v^*_{k+1}, we can use the Helly property to impose a new vertex v^*_k which is adjacent to all v^*_{k+1}, w_k, v'_k, v'_{k-1} and at distance k from a. Continuing this way, we obtain a new vertex v^*_i which is adjacent to all v^*_{i+1}, w_i, v'_i, v'_{i-1} and at distance i from a (for $i = k-1, k-2, \ldots, 1$).

For each edge $u'_i u'_{i+1}$ denote by y_{i+1} a vertex of $H_2^{k,l}$ which forms a triangle with $u'_i u'_{i+1}$. Since vertices d, v^*_{k+1}, u'_l, v'_k are pairwise at distance at most 2 and at distance at most $l+2$ from c, by the Helly property, there must exist a vertex u^*_{l+1} adjacent to d, v^*_{k+1}, u'_l, v'_k and at distance $l+1$ from c. Having vertex u^*_{l+1}, we can use the Helly property to impose a new vertex u^*_l which is adjacent to all u^*_{l+1}, u'_l, u'_{l-1}, y_l and at distance l from c. Continuing this way, we obtain a new vertex u^*_i which is adjacent to all u^*_{i+1}, u'_i, u'_{i-1}, y_i and is at distance i from c (for $i = l-1, l-2, \ldots, 1$). This completes the extension of $H_2^{k,l}$ to $H_3^{k,l}$.

Clearly, $H_3^{k,l}$ obtained from $H_2^{k,l}$ is an isometric subgraph of G. Recall that $H_3^{k,l}$ is a $\{a, b, c, d\}$-distance preserving subgraph of G. We know from Lemma 6 that $H_2^{k,l}$-part of $H_3^{k,l}$ is an isometric subgraph of G. We know also that every pair $x, y \in H_3^{k,l} \setminus H_2^{k,l}$ belongs to a shortest path of G from a to d or from d to c or from a to c passing through a neighbor of d. Finally, every pair x, y with $x \in H_3^{k,l} \setminus H_2^{k,l}$ and $y \in H_2^{k,l}$ belongs to a shortest path of G connecting s with t where $s, t \in \{a, b, c, d\}$. □

Combining Lemmas 3, 4 and 7, we conclude with a tight bound on hyperbolicity with respect to interval thinness in Helly graphs, as well as with a characterization of the case in which the hyperbolicity of a Helly graph realizes the upper bound.

Theorem 2. *For every Helly graph G, $\tau(G) \leq 2hb(G) \leq \tau(G) + 1$. Furthermore, $2hb(G) = \tau(G) + 1$ if and only if $\tau(G)$ is odd and G contains graph H_3^k with $k = \lfloor \frac{\tau(G)}{2} \rfloor$ as an isometric subgraph.*

Corollary 1. *For every Helly graph G, if $\tau(G)$ is even, then $hb(G)$ is an integer and $2hb(G) = \tau(G)$.*

4. Three isometric subgraphs of the King-grid are the only obstructions to a small hyperbolicity in Helly graphs

In this section, we will identify three isometric subgraphs of the King-grid that are responsible for the hyperbolicity of a Helly graph G. These are named H_1^k, H_2^k, H_3^k, and are shown in Fig. 9. We may assume that $hb(G) > 0$ as the structure of any graph with hyperbolicity 0 is well-known; they are exactly the block graphs, i.e., graphs where each biconnected component is a complete graph [26].

The following lemma shows the existence of one of the three isometric subgraphs in a Helly graph G with $hb(G) = k > 0$.

Lemma 8. *Let G be a Helly graph with $hb(G) = k > 0$.*
If $\tau(G) = 2k$ and k is an integer, then G contains H_1^k as an isometric subgraph.
If $\tau(G) = 2k$ and k is a half-integer, then G contains $H_2^{k-\frac{1}{2}}$ as an isometric subgraph.
If $\tau(G) = 2k - 1$, then k is an integer and G contains H_3^{k-1} as an isometric subgraph.

Proof. Let $hb(G) = k > 0$, and let interval $I(x, y)$ realize the maximum thinness, that is there are vertices $z, t \in S_\alpha(x, y)$, for some integer α, such that $d(z, t) = \tau(G)$. By Theorem 2, either $\tau(G) = 2k$ or $\tau(G) = 2k - 1$. If $\tau(G) = 2k - 1$, then by Theorem 2, $\tau(G)$ is odd (thus k is an integer) and G contains H_3^{k-1} as an isometric subgraph. If $\tau(G) = 2k$, then $\tau(G)$ can be even or odd (since k can be a half-integer). Set $\alpha := d(x, t) = d(x, z)$, and $\beta := d(t, y) = d(z, y)$.

Let $\tau(G) = 2k$ be even (thus k is an integer). Clearly $\alpha \geq k$ and $\beta \geq k$, otherwise $d(z, t) < 2k$. By Lemma 2, there is a vertex x' such that $d(x, x') = \alpha - k$, $d(z, x') = k$, and $d(t, x') = k$, and there is a vertex y' such that $d(y, y') = \beta - k$, $d(z, y') = k$, and $d(z, y') = k$. By the triangle inequality, $d(x', y') \leq d(x', z) + d(z, y') = 2k$ and $\alpha + \beta = d(x, y) \leq \alpha - k + d(x', y') + \beta - k \leq \alpha + \beta$. Therefore, $d(x', y') = 2k$ must hold. Then, by Lemma 5, G contains an isometric subgraph H_1^k with $\{x', z, y', t\}$ as corner points.

Let $\tau(G) = 2k$ be odd (thus k is a half-integer). Let $k = p + \frac{1}{2}$ for an integer p. Then $d(z, t) = 2p + 1$. Clearly $\alpha > p$ and $\beta > p$, otherwise $d(z, t) < 2p + 1$. By Lemma 2, there is a triangle $\triangle(x', x_z, x_t)$ such that $d(x, x') = \alpha - p - 1$, $d(x_z, z) = p$, and $d(x_t, t) = p$, and there is a triangle $\triangle(y', y_z, y_t)$ such that $d(y, y') = \beta - p - 1$, $d(y_z, z) = p$, and $d(y_t, t) = p$. By the triangle inequality, $d(x', y') \leq d(x', x_z) + d(x_z, z) + d(z, y_z,) + d(y_z, y') = 2p + 2$ and $\alpha + \beta = d(x, y) \leq \alpha - p - 1 + d(x', y') + \beta - p - 1 = \alpha + \beta$. Therefore, $d(x', y') = 2p + 2$. Since $p = k - \frac{1}{2}$, by Lemma 6, G contains an isometric subgraph $H_2^{k-\frac{1}{2}}$ with $\{x', z, y', t\}$ as corner points. □

Using the previous lemma, we can now characterize Helly graphs G with $hb(G) \leq \delta$ based on three forbidden isometric subgraphs. Whether δ is an integer or a half-integer determines which of the H_1^k, H_2^k, H_3^k graphs are forbidden and the value of k.

Theorem 3. *Let G be a Helly graph and k be a non-negative integer.*
– $hb(G) \leq k$ if and only if G contains no H_2^k as an isometric subgraph.
– $hb(G) \leq k + \frac{1}{2}$ if and only if G contains neither H_1^{k+1} nor H_3^k as an isometric subgraph.

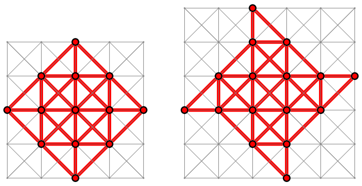

Fig. 13. Forbidden isometric subgraphs for $\frac{3}{2}$-hyperbolic Helly graphs.

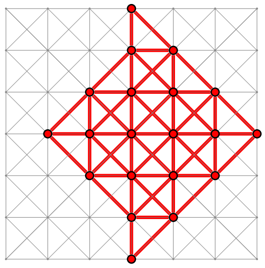

Fig. 14. Forbidden isometric subgraph for 2-hyperbolic Helly graphs.

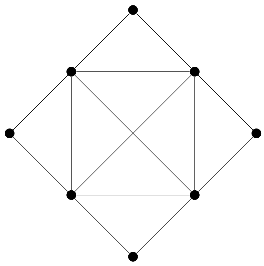

Fig. 15. The graph H_3^0, also known as the 4-sun S_4.

Proof. Assume $hb(G) \leq k$ and that G has H_2^k as an isometric subgraph. It is easy to check that $hb(H_2^k) = k + \frac{1}{2} > k$ (the hyperbolicity realizes on four extreme vertices). As the hyperbolicity of a graph is at least the hyperbolicity of its isometric subgraph, $hb(G) > k$, giving a contradiction.

Assume $hb(G) \leq k+\frac{1}{2}$, and that G has H_1^{k+1} or H_3^k as an isometric subgraph. It is easy to check that $hb(H_1^{k+1}) = k+1 > k+\frac{1}{2}$ and $hb(H_3^k) = k + 1 > k + \frac{1}{2}$ (the hyperbolicity of each realizes on four extreme vertices). As the hyperbolicity of a graph is at least the hyperbolicity of its isometric subgraph, $hb(G) > k + \frac{1}{2}$, giving a contradiction.

For the other direction, assume $hb(G) = \delta$. Then, by Lemma 8, G has one of $H_1^\delta, H_2^{\delta-\frac{1}{2}}, H_3^{\delta-1}$ as an isometric subgraph. Note that, for any integer m, H_1^m is an isometric subgraph of H_2^m and H_1^{m+1}, H_2^m is an isometric subgraph of H_3^m, H_2^{m+1} and H_1^{m+1}, and H_3^m is an isometric subgraph of H_3^{m+1}. If δ is an integer, G contains H_1^δ or $H_3^{\delta-1}$, and hence H_1^{k+1} or H_3^k when $\delta > k + \frac{1}{2}$, as an isometric subgraph. If δ is a half-integer, G contains $H_2^{\delta-\frac{1}{2}}$, and hence H_2^k when $\delta \geq k + \frac{1}{2}$, as an isometric subgraph. □

Theorem 3 can easily be applied to determine the forbidden subgraphs characterizing any δ-hyperbolic Helly graph. The corollaries that follow exemplify this for $\frac{3}{2}$-hyperbolic Helly graphs and 2-hyperbolic Helly graphs.

Corollary 2. *A Helly graph is $\frac{3}{2}$-hyperbolic if and only if it contains neither of graphs from Fig. 13 as an isometric subgraph.*

Corollary 3. *A Helly graph is 2-hyperbolic if and only if it does not contain graph from Fig. 14 as an isometric subgraph.*

To give a few equivalent characterizations of $\frac{1}{2}$-hyperbolic Helly graphs, we will need one more lemma. Let C_4 denote an induced cycle on four vertices. We say that a graph G is C_4-*free* if it does not contain C_4 as an induced subgraph. The graph H_3^0 is also known in the literature as the 4-sun S_4 (see Fig. 15).

Lemma 9 ([21]). *For any C_4-free Helly graph G, every C_4 in G^2 forms in G an isometric subgraph S_4.*

By combining Theorems 2, 3 and Lemma 9, we obtain the following characterization of $\frac{1}{2}$-hyperbolic Helly graphs. One necessary and sufficient condition is that G and G^2 are C_4-free. In fact, in the characterization of any $\frac{1}{2}$-hyperbolic graph [16], there is a similar requirement that every graph power G^i for $i \geq 1$ is C_4-free and one additional graph is C_4-free. We explore this relationship between C_4-free graph powers and the δ-hyperbolicity of any Helly graph in subsequent results presented here.

Corollary 4. *The following statements are equivalent for any Helly graph G:*

(i) G is $\frac{1}{2}$-hyperbolic;
(ii) G has neither C_4 nor S_4 as an isometric subgraph;
(iii) Neither G nor G^2 has an induced C_4;
(iv) $\tau(G) \leq 1$ and G has no S_4 as an isometric subgraph.

The following lemmas describe the three forbidden isometric subgraphs in terms of graph powers.

Lemma 10. *Let G be a Helly graph and k be a non-negative integer. Then G has H_1^{k+1} as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$.*

Proof. Suppose G has H_1^{k+1} as an isometric subgraph. Then, for four extreme vertices x, y, z, t of H_1^{k+1}, we have $d(x, y) = d(y, z) = d(z, t) = d(t, x) = k + 1$ and $d(x, z) = d(y, t) = 2k + 2$. Thus, x, y, z, t, form C_4 in G^ℓ for all $\ell \in [k + 1, 2k + 1]$.

Now, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k + 1, 2k + 1]$. Then, each of $d(x, y), d(y, z), d(z, t)$, $d(t, x)$ is less than or equal to $k+1$, and $d(x, z), d(y, t)$ are greater than or equal to $2k+2$. From these distance requirements, necessarily, $d(x, y) = d(y, z) = d(z, t) = d(t, x) = k + 1$ and $d(x, z) = d(y, t) = 2k + 2$. By Lemma 5, G has isometric H_1^{k+1}. □

A cycle on 4 vertices with one diagonal is called a *diamond*.

Lemma 11. *Let G be a Helly graph and k be a non-negative integer. Then G has H_2^k as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}.*

Proof. Suppose G has H_2^k as an isometric subgraph. Then, for four extreme vertices x, y, z, t of H_2^k, we have $d(x, y) = d(y, z) = d(z, t) = d(t, x) = k + 1$ and $d(x, z) = 2k + 2$ and $d(y, t) = 2k + 1$. Thus, x, y, z, t, form C_4 in G^ℓ for all $\ell \in [k + 1, 2k]$ and form a diamond in G^{2k+1}.

Next, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k + 1, 2k]$ and form a diamond in G^{2k+1}. Then, each of $d(x, y), d(y, z), d(z, t), d(t, x)$ is less than or equal to $k + 1$. Without loss of generality, let yt be the chord of a diamond in G^{2k+1} formed by x, y, z, t. Thus, $d(y, t) \geq 2k + 1$ and $d(x, z) \geq 2k + 2$. From these distance requirements, necessarily, $d(x, y) = d(y, z) = d(z, t) = d(t, x) = k + 1, 2k + 1 \leq d(y, t) \leq 2k + 2$ and $d(x, z) = 2k + 2$. If $d(y, t) = 2k + 2$, then by Lemma 5, G has an isometric H_1^{k+1}, and hence an isometric H_2^k (note that H_1^{k+1} contains an isometric H_2^k). Let now $d(y, t) = 2k + 1$. By Lemma 2, there exist shortest paths $P(x, y)$ and $P(x, t)$ such that the neighbors of x on those paths are adjacent. Similarly, there exist shortest paths $P(z, y)$ and $P(z, t)$ such that the neighbors of z on those paths are adjacent. Thus, x, y, z, t form $\{x, y, z, t\}$-distance preserving subgraph depicted in Fig. 3(b). By Lemma 6, G has an isometric H_2^k. □

The following result generalizes Lemma 9.

Lemma 12. *Let G be a Helly graph and k be a non-negative integer. Then G has H_1^{k+1} or H_3^k as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$ or there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.*

Proof. By Lemma 10, G has H_1^{k+1} as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k + 1, 2k + 1]$. Suppose G has H_3^k as an isometric subgraph. Then, for four extreme vertices x, y, z, t, we have $d(x, y) = d(y, z) = d(z, t) = d(t, x) = k+2$ and $d(x, z) = d(y, t) = 2k+3$. Thus, x, y, z, t form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.

Next, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k + 2, 2k + 2]$. Then, each of $d(x, y), d(y, z), d(z, t)$, and $d(t, x)$ is less than or equal to $k+2$, and each of $d(x, z)$ and $d(y, t)$ is greater than or equal to $2k+3$. Additionally, $d(x, y)$, $d(y, z), d(z, t)$, and $d(t, x)$ must be greater than or equal to $k+1$, since otherwise $d(x, z) < 2k+3$ and $d(y, t) < 2k+3$. Thus, $2k + 3 \leq d(x, z) \leq 2k + 4$ and $2k + 3 \leq d(y, t) \leq 2k + 4$. We consider three cases.

In case 1, let $d(x, z) = d(y, t) = 2k + 4$. Then, necessarily, $d(x, y) = d(y, z) = d(t, z) = d(t, x) = k + 2$. By Lemma 5, G has an isometric H_1^{k+2}. Since H_1^{k+1} is an isometric subgraph of H_1^{k+2}, G has an isometric H_1^{k+1}.

In case 2, let $d(x, z) = 2k + 3$ and $d(y, t) = 2k + 4$. Then $d(x, y) = d(y, z) = d(t, z) = d(t, x) = k + 2$ (otherwise, $d(y, t) < 2k + 4$). As in the proof of Lemma 11, we conclude that G has an isometric H_2^{k+1}. Thus, G has an isometric H_1^{k+1} (recall that H_2^{k+1} contains an isometric H_1^{k+1}).

In case 3, let $d(x, z) = d(y, t) = 2k + 3$. First assume, without loss of generality, that $d(x, y) = k + 1$. Then, necessarily, $d(x, t) = d(y, z) = k + 2$. If also $d(t, z) = k + 1$ then, by Lemma 5, G has an isometric $H_1^{k+1,k+2}$. Since H_1^{k+1} is an isometric

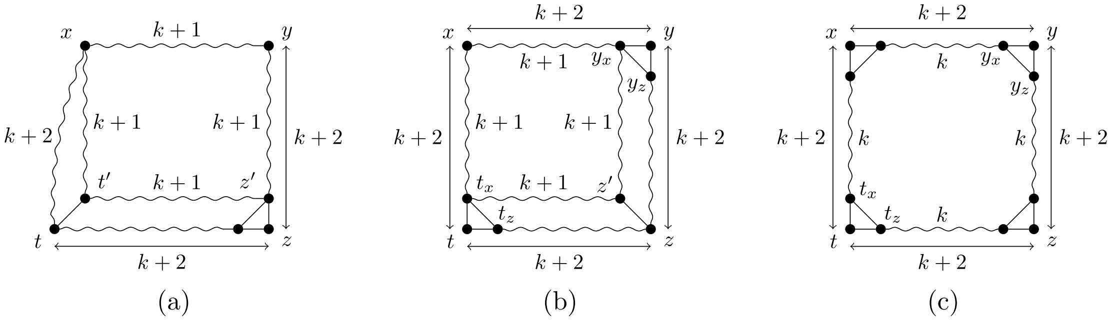

Fig. 16. Illustrations for Case 3 for the proof of Lemma 12.

subgraph of $H_1^{k+1,k+2}$, G has an isometric H_1^{k+1}. If now $d(t,z) = k+2$ then, by Lemma 2 applied to y, z, t, there exist shortest paths $P(z,y)$ and $P(z,t)$ such that the neighbors of z on those paths are adjacent. Let z' be the neighbor of z on $P(z,y)$. We have $d(x,t) = d(t,z') = k+2$, $d(x,y) = d(y,z') = k+1$ and hence $d(x,z') = 2k+2$ as $d(x,z) = 2k+3$. By Lemma 2 applied to x, z', t, there exists a vertex t' adjacent to t such that $d(t',x) = k+1$ and $d(t',z') = k+1$, as shown in Fig. 16(a). Since $d(x,y) = d(x,t') = k+1$ and $d(y,t) = 2k+3$, necessarily $d(y,t') = 2k+2$. By Lemma 5, G has an isometric H_1^{k+1} with x, y, z', t' as corner points.

To finish case 3, it remains to analyze the situation when $d(x,z) = d(y,t) = 2k+3$ and $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+2$. By Lemma 2 applied to x, z, t, there is a triangle $\triangle(t_x, t, t_z)$ such that t_x is the neighbor of t on a shortest (x,t)-path, and t_z is the neighbor of t on a shortest (z,t)-path. Similarly, by Lemma 2 applied to x, z, y, there is a triangle $\triangle(y_x, y, y_z)$ such that y_x is the neighbor of y on a shortest (x,y)-path, and y_z is the neighbor of y on a shortest (z,y) path. From the distance requirements, $2k+1 \le d(t_x, y_x) \le 2k+2$ and $2k+1 \le d(t_z, y_z) \le 2k+2$ (recall that $d(x,z) = d(y,t) = 2k+3$).

If $d(t_x, y_x) = 2k+2$ then, by Lemma 2 applied to y_x, z, t_x, there exists a vertex z' adjacent to z such that $d(z', y_x) = d(z', t_x) = k+1$, as shown in Fig. 16(b). Necessarily, $d(x,z') = 2k+2$ as $d(x,z) = 2k+3$. Now, $d(x,y_x) = d(x,t_x) = d(z',y_x) = d(z',t_x) = k+1$ and $d(y_x,t_x) = d(x,z') = 2k+2$, and we can apply Lemma 5 and get in G an isometric H_1^{k+1} with x, y_x, z', t_x as corner points. Thus, we may assume that $d(t_x, y_x) = 2k+1$. Similarly, we may assume that $d(t_z, y_z) = 2k+1$.

By Lemma 2 applied to t_x, x, y_x, there exist shortest paths $P(y_x, x)$ and $P(t_x, x)$ such that the neighbors of x on those paths are adjacent. By Lemma 2 applied to y_z, z, t_z, there exist shortest paths $P(y_z, z)$ and $P(t_z, z)$ such that the neighbors of z on those paths are adjacent, as shown in Fig. 16(c). Thus, we have constructed an $\{x,y,z,t\}$-distance preserving subgraph depicted in Fig. 3(c). Hence, by Lemma 7, G has an isometric subgraph H_3^k. □

The following result reformulates Theorem 3 in terms of graph powers. It follows directly from Theorem 3, Lemmas 11 and 12. It relates to a result of Coudert and Ducoffe [16], which characterizes any $\frac{1}{2}$-hyperbolic graph by forbidding C_4 in certain graph powers. Here, we give a characterization for any δ-hyperbolic Helly graph, for all values of δ, by forbidding C_4 and the diamond graph in certain graph powers.

Theorem 4. *Let G be a Helly graph and k be a non-negative integer.*
– *$hb(G) \le k$ if and only if there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}.*
– *$hb(G) \le k + \frac{1}{2}$ if and only if there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$, and there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.*

Acknowledgments

We would like to thank anonymous reviewers for many useful suggestions and comments.

References

[1] M. Abu-Ata, F.F. Dragan, Metric tree-like structures in real-world networks: an empirical study, Networks 67 (2016) 49–68.
[2] A. Adcock, B. Sullivan, M. Mahoney, Tree-like structure in large social and information networks, in: ICDM, 2013, pp. 1–10.
[3] H.-J. Bandelt, V. Chepoi, 1-Hyperbolic graphs, SIAM J. Discr. Math. 16 (2003) 323–334.
[4] H.-J. Bandelt, V. Chepoi, Metric graph theory and geometry: a survey, Contemp. Math. 453 (2008) 49–86.
[5] H.-J. Bandelt, H.M. Mulder, Pseudo-modular graphs, Discrete Math. 62 (1986) 245–260.
[6] G. Brinkmann, J. Koolen, V. Moulton, On the hyperbolicity of chordal graphs, Ann. Comb. 5 (2001) 61–69.
[7] J. Chalopin, V. Chepoi, H. Hirai, D. Osajda, Weakly modular graphs and nonpositive curvature, CoRR abs/1409.3892 (2014).
[8] J. Chalopin, V. Chepoi, P. Papasoglu, T. Pecatte, Cop and Robber Game and Hyperbolicity, SIAM J. Discrete Math. 28 (2014) 1987–2007.
[9] C. Chekuri, J. Chuzhoy, Polynomial bounds for the grid-minor theorem, in: STOC, 2014, pp. 60–69.
[10] W. Chen, W. Fang, G. Hu, M.W. Mahoney, On the hyperbolicity of small-world and treelike random graphs, in: K.-M. Chao, T. Sheng Hsu, D.-T. Lee (Eds.), 23rd International Symposium on Algorithms and Computation, ISAAC 2012, in: Lecture Notes in Computer Science, vol. 7676, Springer, 2012, pp. 278–288.

[11] V.D. Chepoi, F.F. Dragan, B. Estellon, M. Habib, Y. Vaxes, Diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs, in: Proceedings of the 24th Annual ACM Symposium on Computational Geometry, SoCG 2008, June 9–11, College Park, Maryland, USA, 2008, pp. 59–68.
[12] V. Chepoi, F.F. Dragan, B. Estellon, M. Habib, Y. Vaxès, Y. Xiang, Additive Spanners and Distance and Routing Labeling Schemes for Hyperbolic Graphs, Algorithmica 62 (2012) 713–732.
[13] V.D. Chepoi, F.F. Dragan, Y. Vaxes, Core congestion is inherent in hyperbolic networks, in: Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, January 16–19, Barcelona, Spain, 2017.
[14] V. Chepoi, B. Estellon, Packing and covering δ-hyperbolic spaces by balls, in: Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques, in: Lecture Notes in Computer Science, vol. 4627, Springer, 2007, pp. 59–73.
[15] N. Cohen, D. Coudert, G. Ducoffe, A. Lancin, Applying clique-decomposition for computing Gromov hyperbolicity, Theoret. Comput. Sci. 690 (2017) 114–139.
[16] D. Coudert, G. Ducoffe, On the recognition of C_4-free and 1/2-hyperbolic graphs, SIAM J. Discr. Math. 28 (2014) 1601–1617.
[17] D. Coudert, G. Ducoffe, N. Nisse, To Approximate Treewidth, Use Treelength!, SIAM J. Discrete Math. 30 (2016) 1424–1436.
[18] B. DasGupta, M. Karpinski, N. Mobasheri, F. Yahyanejad, Node expansions and cuts in Gromov-hyperbolic graphs, CoRR abs/1510.08779 (2015).
[19] E.D. Demaine, M. Hajiaghayi, The bidimensionality theory and its algorithmic applications, Comput. J. 51 (2008) 292–302.
[20] R. Diestel, Graph Theory, Springer, New York, 2000.
[21] F.F. Dragan, Domination in quadrangle-free helly graphs, Cybernet. Systems Anal. 29 (1993) 822–829.
[22] A.W.M. Dress, tight extensions of metric spaces, and the cohomological dimension of certain groups, Adv. Math. 53 (1984) 321–402.
[23] K. Edwards, W.S. Kennedy, I. Saniee, Fast approximation algorithms for p-centres in large *delta*-hyperbolic graphs, CoRR abs/1604.07359 (2016).
[24] D. Eppstein, Diameter and treewidth in minor-closed graph families, Algorithmica 27 (2000) 275–291.
[25] C. Gavoille, O. Ly, Distance labeling in hyperbolic graphs, in: ISAAC, 2005, pp. 171–179.
[26] M. Gromov, Hyperbolic groups, in: Essays in Group Theory, Springer, 1987.
[27] W.-L. Hsu, G. Nemhauser, Easy and hard bottleneck location problems, Discrete Appl. Math. 1 (1979) 209–215.
[28] J.R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1964) 65–76.
[29] E. Jonckheere, P. Lohsoonthorn, F. Bonahon, Scaled gromov hyperbolic graphs, J. Graph Theory 57 (2008) 157–180.
[30] E.A. Jonckheere, M. Lou, F. Bonahon, Y. Baryshnikov, Euclidean versus hyperbolic congestion in idealized versus experimental networks, Internet Math. 7 (2011) 1–27.
[31] W.S. Kennedy, O. Narayan, I. Saniee, On the hyperbolicity of large-scale networks, in: IEEE International Conference on Big Data, Big Data 2016, 2016, pp. 3344–3351.
[32] J.H. Koolen, V. Moulton, Hyperbolic bridged graphs, European J. Combin. 23 (2002) 683–699.
[33] R. Krauthgamer, J.R. Lee, Algorithms on negatively curved spaces, in: IEEE Symposium on Foundations of Computer Science, FOCS, 2006, pp. 119–132.
[34] U. Lang, Injective hulls of certain discrete metric spaces and groups, J. Topol. Anal. 5 (2013) 297–331.
[35] F. de Montgolfier, M. Soto, L. Viennot, Treewidth and hyperbolicity of the internet, in: IEEE Networks Computing and Applications 2011, IEEE, 2011.
[36] O. Narayan, I. Saniee, Large-scale curvature of networks, Phys. Rev. E 84 (2011) 066108.
[37] N. Robertson, P.D. Seymour, Graph minors II: algorithmic aspects of tree-width, J. Algorithms 7 (1986) 309–322.
[38] Y. Shavitt, T. Tankel, Hyperbolic embedding of internet graph for distance estimation and overlay construction, IEEE/ACM Trans. Netw. 16 (2008) 25–36.
[39] K. Verbeek, S. Suri, Metric embedding, hyperbolic space, and social networks, in: Proceedings of the Thirtieth Annual Symposium on Computational Geometry, SoCG'14, 2014, pp. 501–510.
[40] D.B. West, Introduction to Graph Theory, Prentice Hall, 2000.
[41] Y. Wu, C. Zhang, Hyperbolicity and chordality of a graph, Electron. J. Combin. 18 (2011).