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Notes on diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs

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Abstract

We present simple methods for approximating the diameters, radii, and centers of finite sets in δ-hyperbolic geodesic spaces and graphs. We also provide a simple construction of distance approximating trees of δ-hyperbolic graphs G on n vertices with an additive error $O(\delta \log_2 n)$ comparable with that given by M. Gromov.

1 Introduction

Given a finite set S of points of a metric space (X, d), the *diameter* $diam(S)$ of S is the maximum distance between any two points of S. A *diametral pair* of S

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is any pair of points $x, y \in S$ such that $d(x, y) = diam(S)$. For a point $x \in X$, the set $F_S(x)$ of *furthest neighbors* of S consists of all points of S located at the maximum distance from x. The *eccentricity* $ecc_S(x)$ of a point $x \in X$ is the distance from x to any point of $F_S(x)$. The *center* $C(S)$ of S is the set of all points of X having minimum eccentricity; the points of $C(S)$ are called *central points*. The *radius* $rad(S)$ of S is the eccentricity of central points.

It is well-known that the diameter $diam(S)$ of a set S in a tree network T can be found by running the following simple algorithm: pick an arbitrary point or vertex u of T, run a Breadth-First-Search (BFS) starting from u to find $v \in F_S(u)$, then run a second BFS starting from v to find $w \in F_S(v)$. Then $d(v, w) = diam(S)$, i.e., $\{v, w\}$ is a diametral pair of S. To find the center of S it suffices to take the midpoint c of the unique (u, v)-path if T. This shows that $diam(S) = 2rad(S)$ in tree-networks.

In this talk, we will show that this approach can be adapted to provide fast and accurate approximations of the diameter, radius, and center of finite sets S of δ-hyperbolic geodesic spaces and graphs. We show that if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v, w) \geq diam(S) - 2\delta$ and that $rad(S) \leq d(v, w)/2 + 3\delta$. We also prove that the center $C(S)$ of S is contained in the ball of radius 5δ ($5\delta + 1$ for graphs) centered at the midpoint c of any (v, w)-geodesic. This provides a linear time algorithm for computing the center of δ-hyperbolic graphs with uniformly bounded degrees of vertices. We also give a simple linear-time construction of distance approximating trees of δ-hyperbolic graphs $G = (V, E)$ on n vertices with an additive error $O(\delta \log_2 n)$ (comparable with a result of Gromov [3]). For an extended abstract containing all proofs, see [2].

2 Results

2.1 Diameters, radii, and centers

In this subsection we obtain fast approximations for diameters and radii of finite subsets S of δ-hyperbolic spaces (X, d). We start with the analysis in δ-hyperbolic spaces (X, d) of the simple heuristic for computing a diametral pair in trees. Recall, it consists in picking any point u of the space X, finding $v \in F_S(u)$, then finding $w \in F_S(v)$, and returning the pair $\{v, w\}$. This algorithm can be implemented in $O(|S|)$ time if computing the distance between two points in (X, d) can be done in constant time. In particular, this is the case when (X, d) is a model of the hyperbolic plane. On the other hand, in graphs $G = (V, E)$ (and, more generally, networklike spaces) the pair $\{u, v\}$ can be computed in linear $O(|E|)$ time. The following result establishes how well $d(v,w)$ approximates the diameter $diam(S)$ of S :

Proposition 2.1 *For a finite subset S of a δ-hyperbolic space (X,d) and any point $u \in X$, if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v,w) \geq diam(S) - 2\delta$.*

Next result establishes a relationship between diameters and radii of δ-hyperbolic geodesic spaces and graphs.

Proposition 2.2 *For any finite subset S of a geodesic δ-hyperbolic space or δ-hyperbolic graph, we have $diam(S) \geq 2rad(S) - 4\delta$ and $diam(S) \geq 2rad(S) - 4\delta - 1$, respectively.*

Combining the two previous results, we obtain that, if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v,w) \geq 2rad(S) - 6\delta$ for geodesic δ-hyperbolic spaces and $d(v,w) \geq 2rad(S) - 6\delta - 1$ for δ-hyperbolic graphs. This provides a fast approximation of the radius of S :

Proposition 2.3 *For any finite subset S of a δ-hyperbolic geodesic space or graph, we have $rad(S) \leq d(v,w)/2 + 3\delta$ and $rad(S) \leq \lfloor d(v,w)+1)/2 \rfloor + 3\delta$.*

Next result establishes an upper bound on the diameter of the center $C(S)$.

Proposition 2.4 *$diam(C(S)) \leq 4\delta$ for δ-hyperbolic geodesic spaces and $\leq 4\delta + 1$ for δ-hyperbolic graphs.*

In the next result, the point c is the midpoint (a middle vertex in case of graphs) of any geodesic segment $[v,w]$ between the points v and w defined above. Let also c_0 be the point (vertex) of $[v,w]$ located at distance $rad(S)$ from w. Since $rad(S) \geq d(w,c) = d(v,w)/2 \geq rad(S) - 3\delta$, we conclude that for geodesic spaces, $d(c,c_0) \leq 3\delta$ and that c_0 is located on the geodesic $[c,v]$ between c and v (for graphs, $d(c,c_0) \leq 3\delta + 1$).

Proposition 2.5 *The inequalities $ecc(c) \leq rad(S)+5\delta$ and $ecc(c_0) \leq rad(S)+2\delta$ hold for all δ-hyperbolic geodesic spaces and graphs. Moreover $C(S) \subseteq B(c,5\delta)$ for δ-hyperbolic geodesic spaces ($C(G) \subseteq B(c,5\delta+1)$ for δ-hyperbolic graphs).*

2.2 Approximating trees

In this section, we present a simple method which constructs for any δ-hyperbolic graph $G=(V,E)$ with n vertices a distance $O(\delta \log n)$-approximating tree in optimal time $O(|E|)$. A tree $T=(V,F)$ is called a *distance κ-approximating tree* of a graph $G=(V,E)$ if $|d_G(x,y) - d_T(x,y)| \leq \kappa$ for each pair of vertices

$x, y \in V$. The error incurred by our result is slightly weaker (but of the same order) than by a similar result of M. Gromov [3], however the construction of our approximating tree T is simpler and can be done in linear $O(|E|)$ time.

Let $G = (V, E)$ be a connected graph with a distinguished vertex s. A *layering* of G with respect to s is the partition of V into the *spheres* $L^i = \{u \in V : d(s, u) = i\}$, $i = 0, 1, 2, \ldots$ A *layering partition* of G is a partition of each L^i into clusters $L^i_1, \ldots, L^i_{p_i}$ such that two vertices $u, v \in L^i$ belong to the same cluster L^i_j if and only if they can be connected by a path outside the ball $B_{i-1}(s)$ of radius $i-1$ centered at s (this partition has been introduced in [1]). Set $\Lambda_n := 4 + 3\delta + 2\delta \log_2 n$.

Proposition 2.6 *Let L^i_j be a cluster of a layering partition of a graph G with δ-thin geodesic triangles and n vertices, and let $u, v \in L^i_j$. Then $d_G(u, v) \leq \Lambda_n$.*

Let Γ be a graph whose vertex set is the set of all clusters L^i_j in a layering partition of G and two vertices L^i_j and $L^{i'}_{j'}$ are adjacent in Γ if and only if there exist $u \in L^i_j$ and $v \in L^{i'}_{j'}$ such that u and v are adjacent in G. It is shown in [1] that Γ is a tree, called the *layering tree* of G, and that Γ is computable in linear time in the size of G. To construct the tree $T = (V, F)$, for each cluster $C := L^i_j$ we select a vertex v_C of L^{i-1} which is adjacent in G with at least one vertex of C and make v_C adjacent in T to all vertices of C. Since Γ is a tree, T is a tree as well.

Proposition 2.7 *$T = (V, F)$ is a Λ_n-approximating tree for a graph $G = (V, E)$ with δ-thin geodesic triangles and n vertices. In particular, $T = (V, F)$ is a $4\Lambda_n$-approximating tree for a δ-hyperbolic graph.*

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