Spanners in Sparse Graphs

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Abstract. A *t-spanner* of a graph G is a spanning subgraph S in which the distance between every pair of vertices is at most t times their distance in G. If S is required to be a tree then S is called a *tree t-spanner* of G. In 1998, Fekete and Kremer showed that on unweighted planar graphs the *tree t-spanner problem* (the problem to decide whether G admits a tree t-spanner) is polynomial time solvable for $t \leq 3$ and is NP-complete as long as t is part of the input. They also left as an open problem whether the tree t-spanner problem is polynomial time solvable for every fixed $t \geq 4$. In this work we resolve this open problem and extend the solution in several directions. We show that for every fixed t, it is possible in polynomial time not only to decide if a planar graph G has a tree t-spanner, but also to decide if G has a t-spanner of *bounded treewidth*. Moreover, for every fixed values of t and k, the problem, for a given planar graph G to decide if G has a t-spanner of treewidth at most k, is not only polynomial time solvable, but is *fixed parameter tractable* (with k and t being the parameters). In particular, the running time of our algorithm is linear with respect to the size of G. We extend this result from planar to a much more general class of sparse graphs containing graphs of bounded genus. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some vertices of G. We show that the problem of finding a t-spanner of treewidth k is fixed parameter tractable on graphs that do not contain some fixed apex graph as a minor, i.e. on *apex-minor-free graphs*. Graphs of bounded treewidth are sparse graphs and our technique can be used to settle the complexity of the parameterized version of the *sparse t-spanner problem*, where for given t and m one asks if a given n-vertex graph has a t-spanner with at most $n - 1 + m$ edges. Our results imply that the sparse t-spanner problem is fixed parameter tractable on apex-minor-free graphs with t and m being the parameters. Finally we show that the tractability border of the t-spanner problem cannot be extended beyond the class of apex-minor-free graphs. In particular, we prove that for every $t \geq 4$, the problem of finding a tree t-spanner is NP-complete on *K_6-minor-free graphs*. Thus our results are tight, in a sense that the restriction of input graph being apex-minor-free cannot be replaced by H-minor-free for some non-apex fixed graph H.

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1 Introduction

One of the basic questions in the design of routing schemes for communication networks is to construct a spanning network which has two (often conflicting) properties: it should have simple structure and nicely approximate distances of the network. This problem fits in a larger framework of combinatorial and algorithmic problems that are concerned with distances in a finite metric space induced by a graph. An arbitrary metric space (in particular a finite metric defined by a graph) might not have enough structure to exploit algorithmically. A powerful technique that has been successfully used recently in this context is to embed the given metric space in a simpler metric space such that the distances are approximately preserved in the embedding. New and improved algorithms have resulted from this idea for several important problems (see, e.g., [2,7,25]). Tree metrics are a very natural class of simple metric spaces since many algorithmic problems become tractable on them.

Peleg and Ullman [30] suggested the following parameter to measure the quality of a spanner. The spanner S of a graph G has the *stretch factor* t if the distance in S between any two vertices is at most t times the distance between these vertices in G. A *tree* t*-spanner* of a graph G is a spanning tree with a stretch factor t. If we approximate the graph by a tree t-spanner, we can solve the problem on the tree and the solution interpret on the original graph. Unfortunately, not many graph families admit good tree spanners. This motivates the study of sparse spanners, i.e. spanners with a small amount of edges. There are many applications of spanners in various areas; especially, in distributed systems and communication networks. In [30], close relationships were established between the quality of spanners (in terms of stretch factor and the number of spanner edges), and the time and communication complexities of any synchronizer for the network based on this spanner. Another example is the usage of tree t-spanners in the analysis of arrow distributed queuing protocols [14,24]. Sparse spanners are very useful in message routing in communication networks; in order to maintain succinct routing tables, efficient routing schemes can use only the edges of a sparse spanner [31]. We refer to the survey paper of Peleg [27] for an overview on spanners.

In this work we study t-spanners of bounded treewidth (we postpone the definition of treewidth till the next section). Specifically,

PROBLEM: k-Treewidth t-spanner
INSTANCE: A connected graph G and integers k and t.
QUESTION: Is there a t-spanner S of G of treewidth at most k?

Many algorithmic problems are tractable on graphs of bounded treewidth, and a spanner of small treewidth can be used to obtain an approximate solution to a problem on G. Since every connected graph with n vertices and at most $n-1+m$ edges is of treewidth at most $m+1$, we can see this problem as a generalization of tree t-spanner and sparse t-spanner problems.

Related work. Substantial work has been done on the tree t-spanner problem, also known as the minimum stretch spanning tree problem. Cai and Corneil

[6] have shown that, for a given graph G, the problem to decide whether G has a tree t-spanner is NP-complete for any fixed $t \geq 4$ and is linear time solvable for $t = 1, 2$ (the status of the case $t = 3$ is open for general graphs). An $O(\log n)$-approximation algorithm for the minimum value of t for the tree t-spanner problem is due to Emek and Peleg [21]. See the survey of Peleg [27] on more details on this problem and its variants.

The tree t-spanner problem on planar graphs was studied intensively. Fekete and Kremer [22] proved that the tree t-spanner problem on planar graphs is NP-complete (when t is part of the input). They also show that it can be decided in polynomial time whether a given planar graph has a tree 3-spanner. They gave also a polynomial time algorithm for any fixed t that decides for planar graphs with bounded face length whether there is a tree t-spanner. For fixed $t \geq 4$, the complexity of the tree t-spanner problem on planar graphs was left as an open problem [22].

There are several works investigating the complexity of the problem on subclasses of planar graphs. Peleg and Tendler [29] showed that the problem can be solved in polynomial time on outerplanar graphs, and also in the special case of 1-face depth graphs in which no interior vertex has degree 2. Boksberger et al. [4] investigated the problem on grids and subgrids. They presented polynomial time algorithm on grids and $O(OPT^4)$-approximation for subgrids.

Sparse t-spanners were introduced in [28] and [30] and since that time studied extensively. We refer the reader to [18,19,20] for some inapproximability and approximability results for the sparse spanner problem on general graphs. On planar (unweighted) graphs, the problem of determining, for a given n-vertex graph G and integers m and t, if G has a t-spanner with at most $n+m-1$ edges is NP-complete for every fixed $t \geq 5$. (The case $2 \leq t \leq 4$ is open.) [5]. A PTAS for the minimum number of edges for a special case of 2-spanners of 4-connected planar triangulations was obtained in [17].

Recently, a lot of work has been done on parameterized algorithms on planar graphs and more general classes of graphs (we refer e.g. to book [15] for more information on parameterized complexity and algorithms). Alber et al. [1] initiated the study of subexponential parameterized algorithms for the dominating set problem and its different variations. Demaine et al. [12,13] gave a general framework called bidimensionality to design parameterized algorithms for many problems on planar graphs and showed how by making use of this framework to extend results from planar graphs to much more general graph classes including H-minor-free graphs. However, this framework cannot be used directly to solve the k-Treewidth t-spanner problem because the theory of Demaine et al. is strongly based on the assumption that the parameterized problem should be minor or edge contraction closed, which is not the case for spanners. In particular, it is easy to construct an example when by contracting of an edge in a graph G with a t-spanner of treewidth k, one can obtain a graph which does not have such a spanner.

Our results. In this paper we resolve the problem left open in [22] and extend the solution in several directions. Our general technique is combinatorial in nature and is based on the following observation. Let $\mathcal{G}$ be a class of graphs such that for every fixed t and every $G \in \mathcal{G}$, the treewidth of every t-spanner of G is $\Omega(treewidth(G))$. Then as an almost direct corollary of Bodlaender's Algorithm and Courcelle's Theorem (see Section 5 for details), we have that the k-Treewidth t-spanner problem is fixed parameter tractable on $\mathcal{G}$. Our main combinatorial result is the proof that the class of apex-minor-free graphs, which contains planar and bounded genus graphs, is in $\mathcal{G}$.

After preliminary Section 2, we start (Section 3) by proving the combinatorial properties of t-spanners in planar graphs. Our main result here is the proof that every t-spanner of a planar graph of treewidth k has treewidth $\Omega(k/t)$. The proof idea is based on the Robertson et al. theorem [33] on planar graphs excluding a grid as a minor. A technical complication of a direct usage of this theorem is that non-existence of a k-treewidth t-spanner in a minor or a contraction of a graph G does not imply non-existence of a k-treewidth t-spanner in G. This is why we have to work here with walls and topological minors.

It is possible to extend the combinatorial result on t-spanners in planar graphs to apex-minor-free graphs (Section 4). This extension is quite technical and is based on a number of new insights on the structure of apex-minor-free graphs. The main tools here are the structural theorem of Robertson and Seymour characterizing graphs excluding a graph as a minor and the theorem of Demaine and Hajiaghayi on grid-minors in such graphs. We find the study of the k-treewidth t-spanner problem on apex-minor-free graphs worth of efforts because of the following reason. Apex-minor-free graphs form a natural barrier for extension of many parameter/treewidth combinatorial bounds which hold for planar graphs [11]. However, for almost every such a parameter, the class of apex-minor-free graphs is not an algorithmic obstacle, in a sense, that very often it is possible to construct parameterized algorithms for H-minor-free graphs, where H is not necessary an apex graph, see, e.g. [12]. Surprisingly, this is not the case for the t-spanner problem. We show that the result on tractability of the problem on the class of apex-minor-free graphs is tight and cannot be extended further: the problem becomes intractable on H-minor-free graphs, when H is not an apex graph. In particular, for every $t \geq 4$, the problem of finding a tree t-spanner is NP-complete even on *K_6-minor-free graphs*.

Due the space restrictions some proofs are omitted here but they are given in the technical report [16].

2 Preliminaries

Let $G = (V, E)$ be an undirected graph with the vertex set V and edge set E. (We often will use notations $V(G) = V$ and $E(G) = E$.) The *distance* $\text{dist}_G(u, v)$ between vertices u and v of a connected graph G is the length (the number of edges) of a shortest (u, v)-path in G.

Let t be a positive integer. A subgraph S of G, such that $V(S) = V(G)$, is called a *(multiplicative) t-spanner*, if $\text{dist}_S(u, v) \leq t \cdot \text{dist}_G(u, v)$ for every pair of vertices u and v. The parameter t is called the *stretch factor* of S. It is easy to see that the t-spanners can equivalently be defined as follows.

Proposition 1. *Let G be a connected graph, and t be a positive integer. A spanning subgraph S of G is a t-spanner of G if and only if for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \le t$.*

Given an edge $e = (x, y)$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and replace all multiple edges by simple edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is a subgraph of a contraction of G. We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free. For example, the class of planar graphs is a K_5-minor-free graph class. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some vertices of G. A graph class $\mathcal{G}$ is *apex-minor-free* if $\mathcal{G}$ excludes a fixed apex graph H as a minor. If an edge of a graph G is replaced by the path between it's ends then it is said that this edge is *subdivided*. A graph H is a *topological minor* of a graph G, if G contains a subgraph which is isomorphic to a graph obtained from H by subdividing some of its edges.

The (r, s)-*grid* is the Cartesian product of two paths of lengths $r-1$ and $s-1$. The (r, s)-*wall* is a graph W_{rs} with the vertex set $\{(i, j) \colon 1 \le i \le r, 1 \le j \le s\}$ such that two vertices (i, j) and (i', j') are adjacent if and only if either $i = i'$ and $j' \in \{j-1, j+1\}$, or $j = j'$ and $i' = i + (-1)^{i+j}$.

Let W_{rs} be a wall. By P_i^h we denote the shortest path connecting vertices $(i, 1)$ and (i, s), and by P_j^v is denoted the shortest path connecting vertices $(1, j)$ and (r, j) with assumption that, for $j > 1$, P_j^v contains only vertices (x, y) with $x = j-1, j$. We call by the *southern part of* W_{rs} the path P_r^h, and by the *northern part of* W_{rs} the path P_1^h. Similar, the *eastern* and the *western* parts are the paths P_s^v and P_2^v, correspondingly.

If W is obtained by subdivision of edges of W_{rs}, with slightly abusing the notation, we also will be using these terms for the paths obtained by subdivisions from the corresponding paths of W_{rs}.

It is easy to check that if a graph G contains the (r, r)-grid as a minor, then it contains W_{rr} as a topological minor. Also if G contains W_{rr} as a topological minor, then it contains $(r, \lfloor r/2 \rfloor)$-grid as a minor.

A *tree decomposition* of a graph G is a pair (X, U) where U is a tree whose vertices we will call *nodes* and $X = (\{X_i \mid i \in V(U)\})$ is a collection of subsets of $V(G)$ such that i)$\bigcup_{i \in V(U)} X_i = V(G)$, ii)for each edge $(v, w) \in E(G)$, there is an $i \in V(U)$ such that $v, w \in X_i$, and iii)for each $v \in V(G)$ the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of U. The *width* of a tree decomposition $(\{X_i \mid i \in V(U)\}, U)$ equals $\max_{i \in V(U)} \{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G. We use notation $\mathbf{tw}(G)$ to denote the treewidth of a graph G. A tree decomposition with U being a path, is called a *path decomposition* and the pathwidth of G is the minimum width over all path decompositions of G.

We will need the following result which is due to Robertson, Seymour & Thomas [33].

Proposition 2 ([33]). *Every planar graph with no (r, r)-grid as a minor has treewidth $\leq 6r - 5$.*

A *surface* Σ is a compact 2-manifold without boundary (we always consider connected surfaces). A *line* in Σ is a subset homeomorphic to $[0, 1]$ and a (closed) *disc* $\Delta \subseteq \Sigma$ is a subset homeomorphic to $\{(x, y) : x^2 + y^2 \leq 1\}$. An O-arc is a subset of Σ homeomorphic to a circle. Whenever we refer to a *Σ-embedded graph G* we consider a 2-cell embedding of G in Σ. To simplify notations, we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider a graph G embedded in Σ as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$.

3 Planar Graphs

In this section we prove that for every fixed t, a planar graph of large treewidth cannot have a t-spanner of small treewidth.

Theorem 1. *Let G be a planar graph of treewidth k and let S be a t-spanner of G. Then the treewidth of S is $\Omega(k/t)$.*

Proof. We need the following technical claim (the proof can be seen in [16]). Let G be a planar graph embedded in the plane and containing the wall W_{rs} as a topological minor. Let W be a subgraph of G isomorphic to a subdivision of W_{rs}. Let Δ be the disc in the plane which is bordered by the union of the southern, western, northern and eastern parts of W (with exclusion of pendant vertices) and containing W.

Claim 1. *For every $t \leq \min\{s/4, r/2\} - 1$, every t-spanner S of G contains a path connecting the southern and the northern parts of W, and a path connecting the eastern and the western parts of W. Moreover, both these paths are in Δ.*

Proof. Let us prove the claim for the eastern and the western parts of W. Suppose that for some t-spanner S of G there is no path completely inside of Δ connecting the eastern and the western parts of W. Consider the path $P^h_{\lceil r/2 \rceil}$ in the wall. We find the first edge (x, y) in this path (starting from the western part) with the following property: there is a path in $S \cap \Delta$ connecting the eastern part of W and x but there are no such paths for y. Clearly, such an edge has to exist. Let P be a shortest path in S connecting x and y. By the choice of x and y, path P is not entirely in Δ. So it can be divided into three subpaths: the first path P_1 connects x with some vertex u on the border of Δ, the second part P_2

connects u with some vertex v, which also lies on the border of Δ, the third path P_3 connects v and y, and $P_1 \cup P_3 \subset \Delta$. Note that vertex u cannot belong to the eastern part, and vertex v cannot belong to the western part. The length of P is at least northern or the southern part, then $\text{dist}_S(x,u) \geq r/2-1 \geq t$. If v is in the northern or the southern part. then $\text{dist}_S(y,v) \geq r/2-1 \geq t$. If u is in the western part and v is in the eastern part, then $\text{dist}_S(x,u)+\text{dist}_S(y,v) \geq s/2-1 \geq t$. Hence, in all cases, the length of P is at least $t+1$, and S is not a t-spanner. The claim for the northern and southern parts is proved by similar arguments. We have only to consider path $P^v_{\lfloor s/2 \rfloor+1}$ instead of $P^h_{\lceil r/2 \rceil}$. Note also that here we need the requirement $t \leq s/4-1$. □

Set now $r = \lfloor \frac{k+4}{6} \rfloor$ and let S be a t-spanner of G. By Proposition 2, G has an (r,r)-grid as a minor. Thus G has an (r,r)-wall W_{rr} as a topological minor. Wall W_{rr} contains $\lfloor \frac{r}{4t+1} \rfloor$ disjoint $(4t+1,r)$-walls. Let W be a subgraph of G isomorphic to a subdivision of W_{rr}. By applying Claim 1 to each $(4t+1,r)$-wall, we have that there are $\lfloor \frac{r}{4t+1} \rfloor$ vertex disjoint paths in S connecting eastern and western parts of W. By similar arguments, S also contains $\lfloor \frac{r}{4t+1} \rfloor$ vertex disjoint paths connecting southern and northern parts of W. The union of these paths contains $(\lfloor \frac{r}{4t+1} \rfloor, \lfloor \frac{r}{4t+1} \rfloor)$-grid as a minor. So, S contains this grid as a minor, too, and the treewidth of S is at least $\lfloor \frac{r}{4t+1} \rfloor = \lfloor \frac{\lfloor (k+4)/6 \rfloor}{4t+1} \rfloor = \Omega(k/t)$. □

4 Apex-Minor-Free Graphs

In this section, we extended the results of Theorem 1 to graphs with bounded genus and to apex-minor-free graphs.

4.1 Bounded Genus

The Euler genus $\mathbf{eg}(\Sigma)$ of a nonorientable surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The Euler genus $\mathbf{eg}(\Sigma)$ of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. The following extension of Proposition 2 on graphs of bounded genus is due to Demaine et al. [12].

Proposition 3 ([12]). *If G is a graph with treewidth more than $6r(\mathbf{eg}(G)+1)$ which is embeddable on a surface with Euler genus $\mathbf{eg}(G)$, then G has the (r,r)-grid as a minor.*

We also need a result roughly stating that if a graph G with a big wall as a topological minor is embedded on a surface Σ of small genus, then there is a disc in Σ containing a big part of the wall of G. This result is implicit in the work of Robertson and Seymour and there are simpler alternative proofs by Mohar and Thomassen [26,35]. We use the variant of this result from Geelen et al. [23]. Combining this result and Proposition 3, and using the same arguments as in the planar case, we have the following theorem, which proof can be seen in [16].

Theorem 2. *Let G be a graph of treewidth k and Euler genus g, and let S be a t-spanner of G. Then the treewidth of S is $\Omega(\frac{k}{t \cdot g^{3/2}})$.*

4.2 Excluding Apex as a Minor

This extension of Theorems 1 and 2 to apex-minor-free graphs is based on a structural theorem of Robertson and Seymour [32]. Before describing this theorem we need some definitions.

Definition 1 (Clique-Sums). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, and $k \geq 0$ an integer. For $i = 1, 2$, let $W_i \subset V_i$, form a clique of size h and let G'_i be the graph obtained from G_i by removing a set of edges (possibly empty) from the clique $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. We define the h-*clique-sum *of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, as the graph obtained by taking the union of G'_1 and G'_2 by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_1 \oplus G_2$ is called the* join *of the sum.*

Note that some edges of G_1 and G_2 are not edges of G, because it is possible that they were added by clique-sum operation. Such edges are called *virtual*.

We remark that $\oplus$ is not well defined; different choices of G'_i and the bijection F could give different clique-sums. A sequence of h-clique-sums, not necessarily unique, which result in a graph G, is called a *clique-sum decomposition* of G.

Definition 2 (h-nearly embeddable graphs). *Let Σ be a surface with boundary cycles $C_1, \ldots, C_h$, i.e. each cycle C_i is the border of a disc in Σ. A graph G is h-*nearly embeddable *in Σ, if G has a subset X of size at most h, called* apices, *such that there are (possibly empty) subgraphs $G_0, \ldots, G_h$ of $G \setminus X$ such that i)$G \setminus X = G_0 \cup \cdots \cup G_h$, ii)$G_0$ is embeddable in Σ, we fix an embedding of G_0, graphs $G_1, \ldots, G_h$ (called* vortices*) are pairwise disjoint, iii)for $1 \leq \cdots \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V(G_0) \cap V(G_i)$, G_i has a path decomposition (B_{ij}), $1 \leq j \leq m_i$, of width at most h such that a)for $1 \leq i \leq h$ and for $1 \leq j \leq m_i$ we have $u_j \in B_{ij}$, b)for $1 \leq i \leq h$, we have $V(G_0) \cap C_i = \{u_{i_1}, \ldots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \ldots, u_{i_{m_i}}$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).*

The following proposition is known as the Excluded Minor Theorem [32] and is the cornerstone of Robertson and Seymour's Graph Minors theory.

Proposition 4 ([32]). *For every graph H there exists an integer h, depending only on the size of H, such that every graph excluding H as a minor can be obtained by h-clique-sums from graphs that can be h-nearly embedded in a surface Σ in which H cannot be embedded.*

We also need the following result of Demaine and Hajiaghayi [13].

Proposition 5 ([13]). *If G is an H-minor-free graph with treewidth more than k, then G has the $(\Omega(k), \Omega(k))$-grid as a minor (the hidden constants in the Ω notation depend only on the size of H).*

Theorem 3. *Let H be a fixed apex graph. For every t-spanner S of an H-minor-free graph G, the treewidth of S is $\Omega(\mathbf{tw}(G))$. (The hidden constants in the Ω notation depend only on the size of H and t).*

Proof. Due to space restrictions only the sketch of the proof is given here (see [16] for the complete proof). Let G be an H-minor-free graph of treewidth k. It is well known, that for any pair of graphs G_1, G_2, $\mathbf{tw}(G_1 \oplus G_2) \leq \max\{\mathbf{tw}(G_1), \mathbf{tw}(G_2)\}$. Thus, by decomposing G as a clique sum described in Proposition 4, we conclude that there is a summand G' in this clique sum such that a) G' is h-almost embeddable in a surface Σ of genus h; b) the treewidth of G' is at least k. The further proof is performed in two steps. First we prove that Σ contains a closed disc $\Delta' \subset \Sigma$ such that *i)* $G' \cap \Delta'$ contains an $(\Omega(k), \Omega(k))$-wall as a topological minor and *ii)* no vertex of $G' \cap \Delta'$ is adjacent to an apex vertex and to a vertex from a vortex. The proof is based on Proposition 5. In the second step, by extending the arguments used for planar graphs on the wall inside Δ', we prove that every t-spanner of G has a large grid as a minor, and thus has treewidth $\Omega(k)$. ☐

5 Algorithmic Consequences

This section discusses algorithmic consequences of the combinatorial results obtained above. The proof of the following generic algorithmic observation is a combination of known results.

Theorem 4. *Let $\mathcal{G}$ be a class of graphs such that, for every $G \in \mathcal{G}$ and every t-spanner S of G, the treewidth of S is at least $\mathbf{tw}(G) \cdot f_{\mathcal{G}}(t)$, where $f_{\mathcal{G}}$ is the function only of t. Then for every fixed k and t, the existence of a t-spanner of treewidth at most k in $G \in \mathcal{G}$ can be decided in linear time.*

Proof. Let $G \in \mathcal{G}$ be a graph on n vertices and m edges. For given integers k and t, we use Bodlaender's Algorithm [3] to decide in time $O(n+m)$ if $\mathbf{tw}(G) \leq k/f_{\mathcal{G}}(t)$ (the hidden constants in the big-O depend only on k and $f_{\mathcal{G}}(t)$). If Bodlaender's Algorithm reports that $\mathbf{tw}(G) > k/f_{\mathcal{G}}(t)$, then we conclude that G does not have a t-spanner of treewidth at most k. Otherwise (when $\mathbf{tw}(G) \leq k/f_{\mathcal{G}}(t)$), Bodlaender's Algorithm computes a tree decomposition of G of width at most $k/f_{\mathcal{G}}(t)$. Now we want to apply Courcelle's Theorem [8,9], namely that every problem expressible in monadic second order logic (MSOL) can be solved in linear time on graphs of constant treewidth. To apply Courcelle's Theorem (and to finish the proof of our Theorem), we have to show that, for every fixed positive integers k and t, the property that a graph S is a t-spanner of treewidth at most k is expressible in MSOL. It is known that the property that a subgraph S has the treewidth at most k is expressible in MSOL for every fixed k (see, for example, [10]). Since any path is a sequence of adjacent edges, we have that the condition "for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$" can be written as an MSOL formula for every fixed t. By Proposition 1, this yields that "S is a t-spanner of treewidth at most k" is expressible in MSOL. ☐

Theorems 3 and 4 imply the following result, which is the main algorithmic result of this paper. (Let us note that for $k = 1$, Corollary 1 provides the answer to the question of Fekete and Kremer [22].)

Corollary 1. *Let H be a fixed apex graph. For every fixed k and t, the existence of a t-spanner of treewidth at most k in an H-minor-free graph G can be decided in linear time.*

It is easy to see that the treewidth of a connected n-vertex graph with $n+m-1$ edges is at most $m+1$. Since for a fixed m, the property of that a S is a spanning subgraph of G with $n + m - 1$ edges is in MSOL, we have (as in the proof of Theorem 4) that the combination of Theorem 3 with Bodlaender's Algorithm and Courcelle's Theorem implies the following corollary

Corollary 2. *Let H be a fixed apex graph. For every fixed m and t, the existence of a t-spanner with at most $n - 1 + m$ edges in an n-vertex H-minor-free graph G can be decided in linear time.*

It is easy to show that it is not possible to extend Theorem 3 to the class of H-minor free graphs, where H is not necessary an apex graph. For $i \geq 1$, let H_i be a graph obtained by adding to the (i, i)-grid a vertex v and making it adjacent to all vertices of the grid. Each of the graphs H_i, $i \geq 1$, does not contain the complete graph on six vertices K_6 as a minor. The treewidth of H_i is i, but it has a 2-spanner of treewidth one, which is the star with center in v. Thus, Theorem 4 cannot be used on graphs excluding a non-apex graph as a minor. Similar "apex-minor-free barrier" for using combinatorial bounds for parameterized algorithms was observed for other problems (e.g., parameterized dominating set [11]). However, for many of those problems, there are parameterized algorithms for H-minor-free graphs, which are based on dynamic programming over clique-sums of apex-minor-free graphs by making use of Robertson-Seymour structural theorem (Proposition 4), see, e.g. [12]. So, for many parameterized problems, combinatorial "apex-minor-free barrier" can be overcame. Surprisingly, this is not the case for the t-spanner problem. In particular, the tree 4-spanner problem is NP-complete on apex graphs, and since each apex graph is K_6-minor-free, it is NP-complete, for example, for K_6-minor-free graphs.

Note also that for apex graphs the claim of Theorem 3 is not correct. For $i \geq 1$, let H_i be a graph obtained by adding to the (i, i)-grid a vertex v and making it adjacent to all vertices of the grid. The graphs H_i, $i \geq 1$, do not contain the complete graph on six vertices K_6 as a minor. The treewidth of H_i is i, but it has a 2-spanner of treewidth one, which is the star with center in v.

Theorem 5. *For every fixed $t \geq 4$, deciding if an apex graph G has a tree t-spanner is NP-complete.*

Proof. The proof of this result is based on a modification of the reduction of Cai and Corneil [6] adapted for our purposes, and is given in [16]. □

6 Conclusion

We have shown that for fixed k and t, one can decide in linear time if an apex-minor-free graph G has a t-spanner of treewidth at most k. The results we used in our proof, Bodlaender's Algorithm and Courcelle's Theorem, have huge hidden constants in the running time, and thus Corollary 1 is of theoretical interest mainly. Since for K_6-minor-free graphs and $t = 4$ the problem is NP complete, we doubt that it is possible to design fast practical algorithms solving t-spanner problem on apex-minor-free graphs. However, it is likely that on planar graphs and for small values of t, our ideas can be used to design practical algorithms. First of all, instead of using Bodlaender's algorithm, one can use Ratcatcher algorithm of Seymour-Thomas [34] to find exact branchwidth of a planar graph. The running time of the algorithm is cubic, but there is no hidden constants. The second bottleneck of our approach for practical applications is the usage of Courcelle's Theorem. Instead of that, for small values of t, it is more reasonable to construct dynamic programming algorithms that use the properties of planarity and of the problem.

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