Spanners in sparse graphs

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Outline

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   - History and related work
   - Our results
2. **Combinatorial bounds**
   - Planar graphs
   - Graphs of bounded genus
   - Apex-minor-free graphs
3. **Algorithmic consequences**
   - Polynomial cases
   - H-minor-free graphs

t-spanners

Definition (t-spanner)

Let t be a positive integer. A subgraph S of G, such that $V(S) = V(G)$, is called a *(multiplicative) t-spanner*, if $\mathrm{dist}_S(u, v) \leq t \cdot \mathrm{dist}_G(u, v)$ for every pair of vertices u and v. The parameter t is called the *stretch factor* of S.

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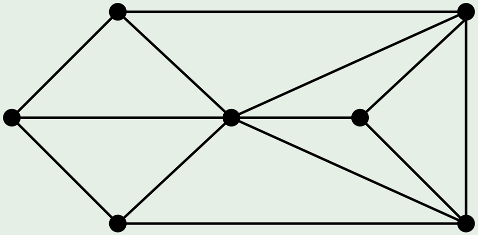
Observation (t-spanner)

Let G be a connected graph, and t be a positive integer. A spanning subgraph S of G is a t-spanner of G if and only if for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$.

Examples of spanners

3 and 2-spanners



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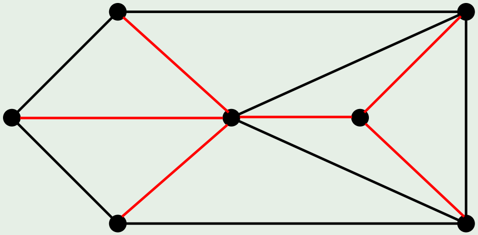

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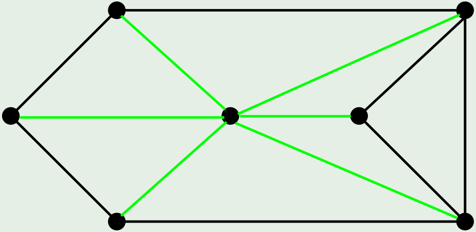

History and related work

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  - It is NP-complete to decide whether there is a tree t-spanner for planar graphs, where t is part of the input.
  - For $t \leq 3$ tree t-spanners can be constructed in polynomial time.

Spanners of bounded treewidth

Problem (k-Treewidth t-spanner)

Instance: A connected graph G and positive integers k and t.
Question: Is there a t-spanner of G of treewidth at most k?

Our results

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- The k-TREEWIDTH t-SPANNER problem is FPT for apex-minor-free graphs
- The k-TREEWIDTH t-SPANNER problem is NP-complete for apex-graphs

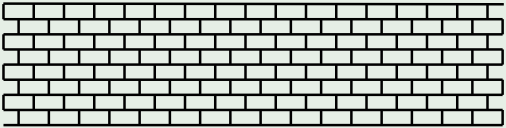
Planar graphs

Theorem (Bounds for planar graphs)

Let G be a planar graph of treewidth k and let S be a t-spanner of G. Then the treewidth of S is $\Omega(k/t)$.

Sketch of the proof

Walls and grids



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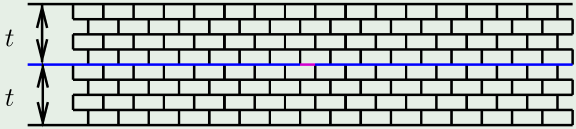

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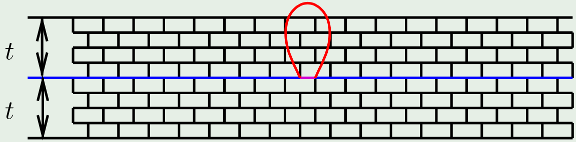

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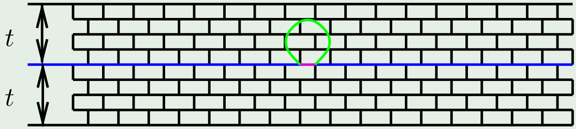

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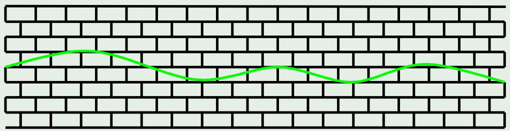

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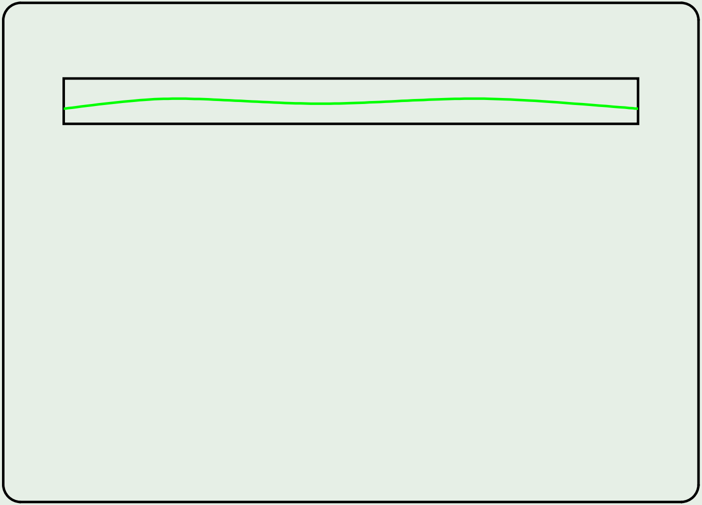

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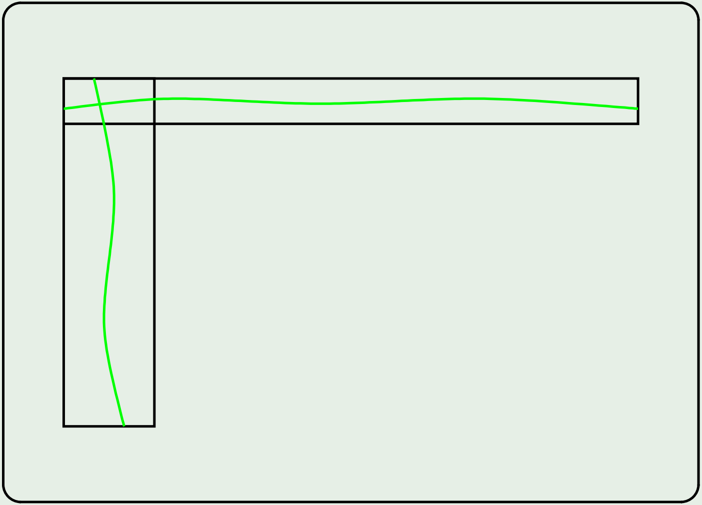

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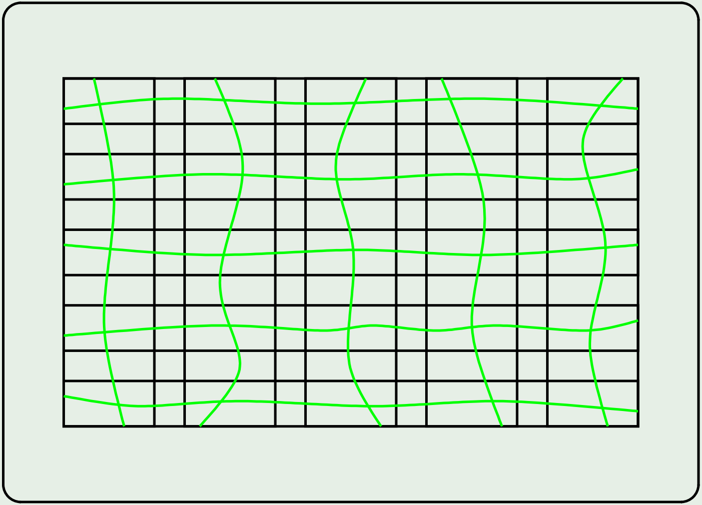

Graphs of bounded genus

Theorem (Bounds for bounded-genus graphs)

Let G be a graph of treewidth k and Euler genus g, and let S be a t-spanner of G. Then the treewidth of S is $\Omega(\frac{k}{t \cdot g^{3/2}})$.

Apex-minor-free graphs

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An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some vertices of G.

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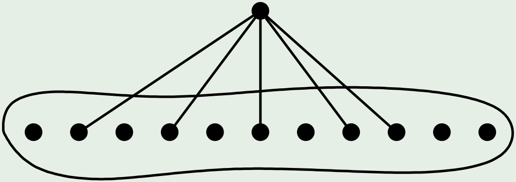

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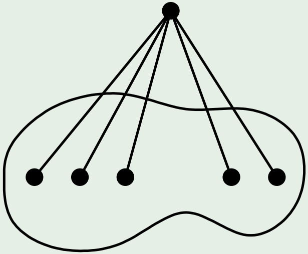
Theorem (Bounds for apex-minor-free graphs)

Let H be a fixed apex graph. For every t-spanner S of an H-minor-free graph G, the treewidth of S is $\Omega(\mathbf{tw}(G))$.

Counterexample for H-minor-free graphs

2-spanner for an apex graph



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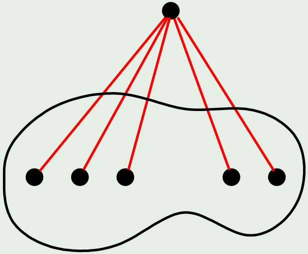

Algorithmic consequences

Theorem (Complexity)

Let $\mathcal{G}$ be a class of graphs such that, for every $G \in \mathcal{G}$ and every t-spanner S of G, the treewidth of S is at least $\mathbf{tw}(G) \cdot f_{\mathcal{G}}(t)$, where $f_{\mathcal{G}}$ is the function only of t. Then for every fixed k and t, the existence of a t-spanner of treewidth at most k in $G \in \mathcal{G}$ can be decided in linear time.

Sketch of the proof

- For given integers k and t, we decide whether $\mathbf{tw}(G) \leq k/f_{\mathcal{G}}(t)$. If $\mathbf{tw}(G) > k/f_{\mathcal{G}}(t)$, then G does not have a t-spanner of treewidth at most k. Otherwise, we construct a tree decomposition of G of width at most $k/f_{\mathcal{G}}(t)$.

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- We apply Courcelle's Theorem: every problem expressible in monadic second order logic (MSOL) can be solved in linear time on graphs of bounded treewidth.
  - The property that a subgraph S has the treewidth at most k is expressible in MSOL for every fixed k.
  - The condition "for every edge (x, y) of G, $\mathrm{dist}_S(x, y) \leq t$" can be written as an MSOL formula for every fixed t.

Algorithmic consequences

Corollary (k-treewidth t-spanners for apex-minor-free graphs)

Let H be a fixed apex graph. For every fixed k and t, the existence of a t-spanner of treewidth at most k in an H-minor-free graph G can be decided in linear time.

Algorithmic consequences

Corollary (k-treewidth t-spanners for apex-minor-free graphs)

Let H be a fixed apex graph. For every fixed k and t, the existence of a t-spanner of treewidth at most k in an H-minor-free graph G can be decided in linear time.

Corollary (Sparse t-spanners for apex-minor-free graphs)

Let H be a fixed apex graph. For every fixed m and t, the existence of a t-spanner with at most $n-1+m$ edges in an n-vertex H-minor-free graph G can be decided in linear time.

Theorem (Apex graphs)

For every fixed $t \geq 4$, deciding if an apex graph G has a tree t-spanner is NP-complete.

Thank you!