Collective Tree Spanners in Graphs with Bounded Genus, Chordality, Tree-Width, or Clique-Width

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Abstract. In this paper we study collective additive tree spanners for special families of graphs including planar graphs, graphs with bounded genus, graphs with bounded tree-width, graphs with bounded clique-width, and graphs with bounded chordality. We say that a graph $G = (V, E)$ *admits a system of* μ *collective additive tree* r*-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that $d_T(x, y) \leq d_G(x, y) + r$. We describe a general method for constructing a "small" system of collective additive tree r-spanners with small values of r for "well" decomposable graphs, and as a byproduct show (among other results) that any weighted planar graph admits a system of $O(\sqrt{n})$ collective additive tree 0–spanners, any weighted graph with tree-width at most $k-1$ admits a system of $k \log_2 n$ collective additive tree 0–spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$–spanners, and any weighted graph with size of largest induced cycle at most c admits a system of $\log_2 n$ collective additive tree $(2\lfloor c/2 \rfloor \mathsf{w})$–spanners and a system of $4 \log_2 n$ collective additive tree $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$–spanners (here, w is the maximum edge weight in G). The latter result is refined for weighted weakly chordal graphs: any such graph admits a system of $4 \log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. Furthermore, based on this collection of trees, we derive a compact and efficient routing scheme for those families of graphs.

1 Introduction

Many combinatorial and algorithmic problems are concerned with the distance d_G on the vertices of a possibly weighted graph $G = (V, E)$. Approximating d_G by a simpler distance (in particular, by tree–distance d_T) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis. Given a graph $G = (V, E)$, a spanning subgraph H is called a *spanner* if H provides a "good" approximation of the distances in G. More formally, for $t \geq 1$, H is called a *multiplicative* t*–spanner* of G [22] if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$. If $r \geq 0$ and $d_H(u, v) \leq d_G(u, v) + r$ for all $u, v \in V$, then H is called an *additive* r*–spanner* of G [21]. The parameters t and r are called, respectively, the *multiplicative* and the *additive stretch factors*. When H is a tree one has a *multiplicative tree* t*-spanner* [4] and an *additive tree* r*-spanner* [23] of G, respectively.

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In this paper, we continue the approach taken in [6, 10, 11, 18] of studying *collective tree spanners*. We say that a graph $G = (V, E)$ *admits a system of μ collective additive tree r-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that $d_T(x, y) \leq d_G(x, y) + r$ (a multiplicative variant of this notion can be defined analogously). Clearly, if G admits a system of μ collective additive tree r-spanners, then G admits an additive r-spanner with at most $\mu\ (n-1)$ edges (take the union of all those trees), and if $\mu = 1$ then G admits an additive tree r-spanner. In particular, we examine the problem of finding "small" systems of collective additive tree r-spanners for small values of r on special classes of graphs such as *planar graphs, graphs with bounded genus, graphs with bounded tree-width, graphs with bounded clique-width, and graphs with bounded chordality.*

Previously, collective tree spanners of particular classes of graphs were considered in [6, 10, 11, 18]. Paper [11] showed that any unweighted chordal graph, chordal bipartite graph or cocomparability graph admits a system of at most $\log_2 n$ collective additive tree 2–spanners. These results were complemented by lower bounds, which say that any system of collective additive tree 1–spanners must have $\Omega(\sqrt{n})$ spanning trees for some chordal graphs and $\Omega(n)$ spanning trees for some chordal bipartite graphs and some cocomparability graphs. Furthermore, it was shown that any unweighted c-chordal graph admits a system of at most $\log_2 n$ collective additive tree $(2\lfloor c/2 \rfloor)$–spanners and any unweighted circular-arc graph admits a system of two collective additive tree 2–spanners. Paper [10] showed that any unweighted AT-free graph (graph without asteroidal triples) admits a system of two collective additive tree 2-spanners and any unweighted graph having a dominating shortest path admits a system of two collective additive tree 3-spanners and a system of five collective additive tree 2-spanners. In paper [6], it was shown that no system of constant number of collective additive tree 1-spanners can exist for unit interval graphs, no system of constant number of collective additive tree r-spanners can exist for chordal graphs and $r \leq 3$, and no system of constant number of collective additive tree r-spanners can exist for weakly chordal graphs and any constant r. On the other hand, [6] proved that any unweighted interval graph of diameter D admits an easily constructable system of $2\log(D-1)+4$ collective additive tree 1-spanners, and any unweighted House-Hole-Domino–free graph with n vertices admits an easily constructable system of at most $2\log_2 n$ collective additive tree 2-spanners. Only paper [18] has investigated (so far) collective (multiplicative) tree spanners in the *weighted graphs* (they were called *tree covers* there). It was shown that any weighted n–vertex planar graph admits a system of $O(\sqrt{n})$ collective multiplicative tree 1-spanners (equivalently, additive tree 0-spanners) and a system of at most $2\log_{3/2} n$ collective multiplicative tree 3–spanners.

One of the motivations to introduce this new concept steams from the problem of designing compact and efficient routing schemes in graphs. In [13, 25], a shortest path routing labeling scheme for trees of arbitrary degree and diameter is described that assigns each vertex of an n-vertex tree a $O(\log^2 n/\log\log n)$-

bit label. Given the label of a source vertex and the label of a destination, it is possible to compute in constant time, based solely on these two labels, the neighbor of the source that heads in the direction of the destination. Clearly, if an n-vertex graph G admits a system of μ collective additive tree r-spanners, then G admits a routing labeling scheme of deviation (i.e., additive stretch) r with addresses and routing tables of size $O(\mu \log^2 n / \log\log n)$ bits per vertex. Once computed by the sender in μ time (by choosing for a given destination an appropriate tree from the collection to perform routing), headers of messages never change, and the routing decision is made in constant time per vertex (for details see [10, 11]).

Our results. In this paper we generalize and refine the method of [11] for constructing a "small" system of collective additive tree r-spanners with small values of r to weighted and larger families of "well" decomposable graphs. We define a large class of graphs, called (α,γ,r)*–decomposable*, and show that any weighted (α,γ,r)–decomposable graph G with n vertices admits a system of at most $\gamma \log_{1/\alpha} n$ collective additive tree $2r$–spanners. Then, we show that all weighted planar graphs are $(2/3, \sqrt{6n}, 0)$–decomposable, all weighted graphs with genus at most g are $(2/3, O(\sqrt{gn}), 0)$–decomposable, all weighted graphs with tree-width at most $k-1$ are $(1/2, k, 0)$–decomposable, all weighted graphs with clique-width at most k are $(2/3, k, \mathsf{w})$–decomposable, all weighted graphs with size of largest induced cycle at most c are $(1/2, 1, \lfloor c/2 \rfloor \mathsf{w})$-decomposable, $(1/2, 6, \lfloor (c+2)/3 \rfloor \mathsf{w})$-decomposable and $(1/2, 4, (\lfloor c/3 \rfloor + 1)\mathsf{w})$-decomposable, and all weighted weakly chordal graphs are $(1/2, 4, \mathsf{w})$-decomposable. Here and in what follows, w denotes the maximum edge weight in G, i.e., $\mathsf{w} := max\{w(e) : e \in E(G)\}$.

As a consequence, we obtain that any weighted planar graph admits a system of $O(\sqrt{n})$ collective additive tree 0–spanners, any weighted graph with genus at most g admits a system of $O(\sqrt{gn})$ collective additive tree 0–spanners, any weighted graph with tree-width at most $k-1$ admits a system of $k \log_2 n$ collective additive tree 0–spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$–spanners, any weighted graph with size of largest induced cycle at most c admits a system of $\log_2 n$ ($6 \log_2 n$ and $4 \log_2 n$) collective additive tree $(2\lfloor c/2 \rfloor \mathsf{w})$–spanners (respectively, $(2\lfloor (c+2)/3 \rfloor \mathsf{w})$–spanners and $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$–spanners), and any weighted weakly chordal graph admits a system of $4 \log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. Furthermore, based on this collection of trees, we derive compact and efficient routing schemes for those families of graphs.

Basic notions and notation. All graphs occurring in this paper are connected, finite, undirected, loopless and without multiple edges. Our graphs can have (non-negative) weights on edges, $w(e)$, $e \in E$, unless otherwise is specified. In a weighted graph $G = (V, E)$ the *distance* $d_G(u, v)$ between the vertices u and v is the length of a shortest path connecting u and v. If the graph is unweighted then, for convenience, each edge has weight 1.

The (*open*) *neighborhood* of a vertex u in G is $N(u) = \{v \in V : uv \in E\}$ and the *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. Define the layers of G with respect to a vertex u as follows: $L_i(u) = \{x \in V : x$ can be connected to u by a path with i edges but not by a path with $i-1$ edges}, $i = 0, 1, 2, \ldots$. In a path $P = (v_0, v_1, \ldots, v_l)$ between vertices v_0 and v_l of G, vertices $v_1, \ldots, v_{l-1}$ are called *inner vertices*. Let r be a non-negative real number. A set $D \subseteq V$ is called an *r-dominating set* for a set $S \subseteq V$ of a graph G if $d_G(v, D) \le r$ holds for any $v \in S$.

A tree-decomposition [24] of a graph G is a tree T whose nodes, called *bags*, are subsets of $V(G)$ such that: 1) $\bigcup_{X \in V(T)} X = V(G)$; 2) for all $\{u, v\} \in E(G)$, there exists $X \in V(T)$ such that $u, v \in X$; and 3) for all $X, Y, Z \in V(T)$, if Y is on the path from X to Z in T then $X \cap Z \subseteq Y$. The *width* of a tree-decomposition is one less than the maximum cardinality of a bag. Among all the tree-decompositions of G, let T be the one with minimum *width*. The width of T is called the *tree-width* of the graph G and is denoted by $tw(G)$. We say that *G has bounded tree-width* if $tw(G)$ is bounded by a constant. It is known that the tree-width of an outerplanar graph and of a series-parallel graph is at most 2 (see, e.g., [19]).

A related notion to tree-width is *clique-width*. Based on the following operations on vertex-labeled graphs, namely (1) creation of a vertex labeled by integer l, (2) disjoint union, (3) join between all vertices with label i and all vertices with label j for $i \neq j$, and (4) relabeling all vertices of label i by label j, the notion of *clique-width* $cw(G)$ of a graph G is defined in [12] as the minimum number of labels which are necessary to generate G by using these operations. Clique-width is a complexity measure on graphs somewhat similar to tree-width, but more powerful since every set of graphs with bounded tree-width has bounded clique-width [7] but not conversely (cliques have clique-width 2 but unbounded tree-width). It is well-known that the clique-width of a cograph is at most 2 and the clique-width of a distance-hereditary graph is at most 3 (see [17]).

The *chordality* of a graph G is the size of the largest (in the number of edges) induced cycle of G. Define *c-chordal graphs* as the graphs with chordality at most c. Then, the well-known chordal graphs are exactly the 3-chordal graphs. An induced cycle of G of size at least 5 is called a *hole*. The complement of a hole is called an *anti-hole*. A graph G is *weakly chordal* if it has neither holes nor anti-holes as induced subgraphs. Clearly, weakly chordal graphs and their complements are 4-chordal. A *cograph* is a graph having no induced paths on 4 vertices (P_4s).

The *genus* of a graph G is the smallest integer g such that G embeds in a surface of genus g without edge crossings. Planar graphs can be embedded on a sphere, hence $g = 0$ for them. A planar graph is *outerplanar* if all its vertices belong to its outerface.

2 (α, γ, r)-Decomposable Graphs and Their Collective Tree Spanners

Let α be a positive real number smaller than 1, γ be a positive integer and r be a non-negative real number. We say that an n-vertex graph $G = (V, E)$ is (α, γ, r)*–decomposable* if there is a separator $S \subseteq V$, such that the following three conditions hold:

Balanced Separator condition - the removal of S from G leaves no connected component with more than αn vertices;

Bounded r-Dominating Set condition - there exists a set $D \subseteq V$ such that $|D| \leq \gamma$ and for any vertex $u \in S$, $d_G(u, D) \leq r$ (we say that D *r-dominates* S);

Hereditary Family condition - each connected component of the graph, obtained from G by removing vertices of S, is also an (α, γ, r)–decomposable graph.

Note that, by definition, any graph having an r-dominating set of size at most γ is (α, γ, r)–decomposable, for any positive $\alpha < 1$.

Using these three conditions, one can construct for any (α, γ, r)-decomposable graph G a *(rooted) balanced decomposition tree* $\mathcal{BT}(G)$ as follows. If G has an r-dominating set of size at most γ, then $\mathcal{BT}(G)$ is a one node tree. Otherwise, find a balanced separator S with bounded r-dominating set in G, which exists according to the first and second conditions. Let $G_1, G_2, \ldots, G_p$ be the connected components of the graph $G \setminus S$ obtained from G by removing vertices of S. For each graph G_i $(i = 1, \ldots, p)$, which is (α, γ, r)-decomposable by the Hereditary Family condition, construct a balanced decomposition tree $\mathcal{BT}(G_i)$ recursively, and build $\mathcal{BT}(G)$ by taking S to be the root and connecting the root of each tree $\mathcal{BT}(G_i)$ as a child of S. Clearly, the nodes of $\mathcal{BT}(G)$ represent a partition of the vertex set V of G into *clusters* $S_1, S_2, \ldots, S_q$, each of them having in G an r-dominating set of size at most γ. For a node X of $\mathcal{BT}(G)$, denote by $G(\downarrow X)$ the (connected) subgraph of G induced by vertices $\cup\{Y : Y$ is a descendent of X in $\mathcal{BT}\}$ (here we assume that X is a descendent of itself).

It is easy to see that a balanced decomposition tree $\mathcal{BT}(G)$ of a graph G with n vertices and m edges has depth at most $\log_{1/\alpha} n$, which is $O(\log_2 n)$ is α is a constant. Moreover, assuming that a special balanced separator (mentioned above) can be found in polynomial, say $p(n)$, time, the tree $\mathcal{BT}(G)$ can be constructed in $O((p(n) + m) \log_{1/\alpha} n)$ total time.

Consider now two arbitrary vertices x and y of an (α, γ, r)-decomposable graph G and let $S(x)$ and $S(y)$ be the nodes of $\mathcal{BT}(G)$ containing x and y, respectively. Let also $NCA_{\mathcal{BT}(G)}(S(x), S(y))$ be the nearest common ancestor of nodes $S(x)$ and $S(y)$ in $\mathcal{BT}(G)$ and $(X_0, X_1, \ldots, X_t)$ be the path of $\mathcal{BT}(G)$ connecting the root X_0 of $\mathcal{BT}(G)$ with $NCA_{\mathcal{BT}(G)}(S(x), S(y)) = X_t$ (in other words, $X_0, X_1, \ldots, X_t$ are the common ancestors of $S(x)$ and $S(y)$). Clearly, any path $P^G_{x,y}$, connecting vertices x and y in G, contains a vertex from $X_0 \cup X_1 \cup \cdots \cup X_t$. Let $SP^G_{x,y}$ be a shortest path of G connecting vertices x and y, and let X_i be the node of the path $(X_0, X_1, \ldots, X_t)$ with the smallest index such that $SP^G_{x,y} \cap X_i \neq \emptyset$ in G. Then, it is easy to show that $d_G(x, y) = d_{G'}(x, y)$, where $G' := G(\downarrow X_i)$.

Let D_i be an r-dominating set of X_i in $G' = G(\downarrow X_i)$ of size at most γ. For the graph G', consider a set of $|D_i|$ *Shortest-Path-trees (SP-trees)* $\mathcal{T}(D_i)$, each rooted at a (different) vertex from D_i. Then, there is a tree $T' \in \mathcal{T}(D_i)$ which has the following distance property with respect to those vertices x and y.

Lemma 1. *For vertices $x, y \in G(\downarrow X_i)$, there exits a tree $T' \in \mathcal{T}(D_i)$ such that $d_{T'}(x, y) \leq d_G(x, y) + 2r$.*

Let now $B^i_1, \ldots, B^i_{p_i}$ be the nodes on depth i of the tree $\mathcal{BT}(G)$ and let $D^i_1, \ldots, D^i_{p_i}$ be the corresponding r-dominating sets. For each subgraph $G^i_j := G(\downarrow B^i_j)$ of G ($i = 0, 1, ..., depth(\mathcal{BT}(G)$, $j = 1, 2, \ldots, p_i$), denote by $\mathcal{T}^i_j$ the set of SP-trees of graph G^i_j rooted at the vertices of D^i_j. Thus, for each G^i_j, we construct at most γ Shortest-Path-trees. We call them *local subtrees*. Lemma 1 implies

Theorem 1. *Let G be an (α, γ, r)-decomposable graph, $\mathcal{BT}(G)$ be its balanced decomposition tree and $\mathcal{LT}(G) = \{T \in \mathcal{T}^i_j : i = 0, 1, \ldots, depth(\mathcal{BT}(G)), j = 1, 2, \ldots, p_i\}$ be its set of local subtrees. Then, for any two vertices x and y of G, there exists a local subtree $T' \in \mathcal{T}^{i'}_{j'} \subseteq \mathcal{LT}(G)$ such that $d_{T'}(x, y) \leq d_G(x, y) + 2r$.*

This theorem implies two import results for the class of (α, γ, r)-decomposable graphs. Let G be an (α, γ, r)-decomposable graph with n vertices and m edges, $\mathcal{BT}(G)$ be its balanced decomposition tree and $\mathcal{LT}(G)$ be the family of its local subtrees (defined above). Consider a graph H obtained by taking the union of all local subtrees of G (by putting all of them together), i.e.,

$$H := \bigcup\{T : T \in \mathcal{T}^i_j \subseteq \mathcal{LT}(G)\} = (V, \cup\{E(T) : T \in \mathcal{T}^i_j \subseteq \mathcal{LT}(G)\}).$$

Clearly, H is a spanning subgraph of G and for any two vertices x and y of G, $d_H(x, y) \leq d_G(x, y) + 2r$ holds. Also, since for any level i ($i = 0, 1, \ldots,$ $depth(\mathcal{BT}(G))$) of balanced decomposition tree $\mathcal{BT}(G)$, the corresponding graphs $G^i_1, \ldots, G^i_{p_i}$ are pairwise vertex-disjoint and $|\mathcal{T}^i_j| \leq \gamma$ ($j = 1, 2, \ldots, p_i$), the union $\bigcup\{T : T \in \mathcal{T}^i_j, j = 1, 2, \ldots, p_i\}$ has at most $\gamma(n-1)$ edges. Therefore, H has at most $\gamma(n-1)\log_{1/\alpha} n$ edges in total. Thus, we have proven the following result.

Theorem 2. *Any (α, γ, r)-decomposable graph G with n vertices admits an additive $2r$-spanner with at most $\gamma(n-1)\log_{1/\alpha} n$ edges.*

Let $\mathcal{T}^i_j := \{T^i_j(1), T^i_j(2), \ldots, T^i_j(\gamma-1), T^i_j(\gamma)\}$ be the set of SP-trees of graph G^i_j rooted at the vertices of D^i_j. Here, if $p := |D^i_j| < \gamma$ then we can set $T^i_j(k) := T^i_j(p)$ for any k, $p+1 \leq k \leq \gamma$. By arbitrarily extending each forest $\{T^i_1(q), T^i_2(q), \ldots, T^i_{p_i}(q)\}$ ($q \in \{1, \ldots, \gamma\}$) to a spanning tree $T^i(q)$ of the graph G, for each level i ($i = 0, 1, \ldots, depth(\mathcal{BT}(G))$) of the decomposition tree $\mathcal{BT}(G)$, we can construct at most γ spanning trees of G. Totally, this will result into at most γ $depth(\mathcal{BT}(G))$ spanning trees $\mathcal{T}(G) := \{T^i(q) : i = 0, 1, \ldots, depth(\mathcal{BT}(G)), q = 1, \ldots, \gamma\}$ of the original graph G. Thus, from Theorem 1, we have the following.

Theorem 3. *Any (α, γ, r)-decomposable graph G with n vertices admits a system $\mathcal{T}(G)$ of at most $\gamma\log_{1/\alpha} n$ collective additive tree $2r$-spanners.*

Corollary 1. *Every (α, γ, r)-decomposable graph G with n vertices admits a routing labeling scheme of deviation $2r$ with addresses and routing tables of size $O(\gamma\log_{1/\alpha} n \log^2 n/\log\log n)$ bits per vertex. Once computed by the sender in $\gamma\log_{1/\alpha} n$ time, headers never change, and the routing decision is made in constant time per vertex.*

3 Particular Classes of (α, γ, r)-Decomposable Graphs

Graphs having balanced separators of bounded size. Here we consider graphs that have balanced separators of bounded size.

To see that planar graphs are $(2/3, \sqrt{6n}, 0)$-decomposable, we recall the following Separator Theorem for planar graphs from [20] (see also [8]) : *The vertices of any n-vertex planar graph G can be partitioned in $O(n)$ time into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has more than 2/3n vertices, and C contains no more than $\sqrt{6n}$ vertices.* Obviously, every connected component of $G \setminus C$ is still a planar graph. The Separator Theorem for planar graphs was extended in [9, 14] to bounded genus graphs: *a graph G with genus at most g admits a separator C of size $O(\sqrt{gn})$ such that any connected component of $G \setminus C$ contains at most $2n/3$ vertices.* Evidently, each connected component of $G \setminus C$ has genus bounded by g, too. Hence, the following results follow.

Theorem 4. *Every n–vertex planar graph is $(2/3, \sqrt{6n}, 0)$-decomposable. Every n–vertex graph with genus at most g is $(2/3, O(\sqrt{gn}), 0)$-decomposable.*

There is another extension of the Separator Theorem for planar graphs, namely, to the graphs with an excluded minor [2]. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. By an H-minor one means a minor of G isomorphic to H. Thus the Pontryagin-Kuratowski-Wagner Theorem asserts that planar graphs are those without K_5 and $K_{3,3}$ minors. The following result was proven in [2]: *Let G be an n-vertex graph and H be an h-vertex graph. If G has no H-minor, then the vertices of G can be partitioned into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has more than 2/3n vertices, and C contains no more than $\sqrt{h^3 n}$ vertices. Furthermore A, B, C can be found in $O(\sqrt{hn}(n+m))$ time, where m is the number of edges in G.*

Since induced subgraphs of an H-minor free graph are H-minor free, we conclude.

Theorem 5. *Let G be an n-vertex graph and H be an h-vertex graph. If G has no H-minor, then G is $(2/3, \sqrt{h^3 n}, 0)$-decomposable.*

Note that, any shortest path routing labeling scheme in n-vertex planar graphs requires at least $\Omega(\sqrt{n})$-bit labels [1]. Hence, by Corollary 1, there must exist n-vertex planar graphs, for which any system of collective additive tree 0-spanners needs to have at least $\Omega(\sqrt{n} \log\log n / \log^2 n)$ trees.

Now we turn to graphs with bounded tree-width. The following theorem is true (proof is omitted).

Theorem 6. *Every graph with tree-width at most k is $(1/2, k+1, 0)$-decomposable.*

Table 1 summarizes the results on collective additive tree spanners of graphs having balanced separators of bounded size. The results are obtained by combining Theorem 3 with Theorems 4, 5 and 6. Note that, for planar graphs, the number of trees in the collection is at most $O(\sqrt{n})$ (not $\sqrt{6n}\log_{3/2} n$ as would follow from Theorem 3). This number can be obtained by solving the recurrent formula $\mu(n) = \sqrt{6n} + \mu(2/3n)$. Similar argument works for other two families of graphs.

Table 1. Collective additive tree spanners of n-vertex graphs having balanced separators of bounded size

Graph class	Number of trees in the collection, μ	additive stretch factor, r
Planar graphs	$O(\sqrt{n})$	0
Graphs with genus g	$O(\sqrt{gn})$	0
Graphs without an h-vertex minor	$O(\sqrt{h^3n})$	0
Graphs with tree-width $k-1$	$k\log_2 n$	0

We conclude this subsection with a lower bound, which follows from a result in [6]. Recall that all outerplanar graphs have tree-width at most 2.

Observation 1. it No system of constant number of collective additive tree r-spanners can exist for outerplanar graphs, for any constant $r \geq 0$.

Graphs with bounded clique-width. Here we will prove that each graph with clique-width at most k is $(2/3, k, \mathsf{w})$-decomposable. Recall that w denotes the maximum edge weight in a graph G, i.e., $\mathsf{w} := max\{w(e) : e \in E(G)\}$.

Theorem 7. *Every graph with clique-width at most* k *is* $(2/3, k, \mathsf{w})$*-decomposable.*

Proof. It was shown in [3] that the vertex set V of any graph $G = (V, E)$ with n vertices and clique-width $cw(G)$ at most k can be partitioned (in polynomial time) into two subsets A and $B := V \setminus A$ such that both A and B have no more than $2/3n$ vertices and A can be represented as the disjoint union of at most k subsets $A_1, \ldots, A_k$ (i.e., $A = A_1\dot{\cup}\ldots\dot{\cup}A_k$), where each A_i ($i \in \{1, \ldots, k\}$) has the property that any vertex from B is either adjacent to all $v \in A_i$ or to no vertex in A_i. Using this, we form a balanced separator S of G as follows. Initially set $S := \emptyset$, and in each subset A_i, arbitrarily choose a vertex v_i. Then, if $N(v_i) \cap B \neq \emptyset$, put v_i and $N(v_i) \cap B$ into S. Since for any edge $ab \in E$ with $a \in A$ and $b \in B$, vertex b must belong to S, we conclude that S is a separator of G, separating $A \setminus S$ from $B \setminus S$. Moreover, each connected component of $G \setminus S$ lies entirely either in A or in B and therefore has at most $2/3n$ vertices. By construction of S, any vertex $u \in B \cap S$ is adjacent to a vertex from $A' := A \cap S$. As $|A'| \leq k$ and w is an upper bound on any edge weight, we deduce that A' w-dominates S in G. Thus, S is a balanced separator of G and is w-dominated by a set A' of cardinality at most k. To finish the proof, it remains to recall that induced subgraphs of a graph with clique-width at most k have clique-width at most k, too (see, e.g., [7]), and therefore, by induction, the connected components of $G \setminus S$ induce $(2/3, k, \mathsf{w})$-decomposable graphs.

Combining Theorem 7 with the results of Section 2, we obtain

Corollary 2. *Any graph with n vertices and clique-width at most k admits a system of at most $k \log_{3/2} n$ collective additive tree* $2\mathsf{w}$*-spanners, and such a system of spanning trees can be found in polynomial time.*

To complement the above result, we give the following lower bound.

Observation 2. it There are (infinitely many) unweighted n-vertex graphs with clique-width at most 2 for which any system of collective additive tree 1-spanners will need to have at least $\Omega(n)$ spanning trees.

Graphs with bounded chordality. Here we consider the class of c-chordal graphs and its subclasses. Proofs of all results of this subsection are omitted.

Theorem 8. *Every n-vertex c-chordal graph is* $(1/2, 1, \lfloor c/2 \rfloor \mathsf{w})$*-decomposable,* $(1/2, 4, (\lfloor c/3 \rfloor + 1)\mathsf{w})$*-decomposable and* $(1/2, 6, \lfloor (c+2)/3 \rfloor \mathsf{w})$*-decomposable.*

Corollary 3. *Every n-vertex c-chordal graph admits a system of at most* $\log_2 n$ *collective additive tree* $(2\lfloor c/2 \rfloor \mathsf{w})$*-spanners, a system of at most* $4 \log_2 n$ *collective additive tree* $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$*-spanners and a system of at most* $6 \log_2 n$ *collective additive tree* $(2(\lfloor (c+2)/3 \rfloor \mathsf{w})$*-spanners. Moreover, such systems of spanning trees can be constructed in polynomial time.*

These results can be refined for 4-chordal graphs and weakly chordal graphs.

Theorem 9. *Every 4-chordal graph is* $(1/2, 6, \mathsf{w})$*-decomposable. Every 4-chordal graph not containing* $\overline{C}_6$ *as an induced subgraph is* $(1/2, 4, \mathsf{w})$*-decomposable.*

Corollary 4. *Any n-vertex 4-chordal graph admits a system of at most* $6 \log_2 n$ *collective additive tree* $2\mathsf{w}$*-spanners. Any n-vertex 4-chordal graph not containing* $\overline{C}_6$ *as an induced subgraph (in particular, any weakly chordal graph) admits a system of at most* $4 \log_2 n$ *collective additive tree* $2\mathsf{w}$*-spanners. Moreover, such systems of spanning trees can be constructed in polynomial time.*

Note that the class of weakly chordal graphs properly contains such known classes of graphs as interval graphs, chordal graphs, chordal bipartite graphs, permutation graphs, trapezoid graphs, House-Hole-Domino–free graphs, distance-hereditary graphs and many others. Hence, the results of this subsection generalize some known results from [6, 11]. We recall also that, as it was shown in [6], no system of constant number of collective additive tree r-spanners can exist for unweighted weakly chordal graphs for any constant $r \geq 0$.

Corollary 5. *Any n-vertex 4-chordal graph admits an additive* $2\mathsf{w}$*-spanner with at most* $O(n \log n)$ *edges. Moreover, such a sparse spanner can be constructed in polynomial time.*

The last result improves and generalizes the known results from [5, 11, 22] on sparse spanners of unweighted chordal graphs.

In full version, we discuss also implication of these results to designing compact routing labeling schemes for graphs under consideration.

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