Collective Additive Tree Spanners of Homogeneously Orderable Graphs

[Extended Abstract]

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Abstract. In this paper we investigate the (*collective*) *tree spanners problem* in homogeneously orderable graphs. This class of graphs was introduced by A. Brandstädt et al. to generalize the dually chordal graphs and the distance-hereditary graphs and to show that the Steiner tree problem can still be solved in polynomial time on this more general class of graphs. In this paper, we demonstrate that every n-vertex homogeneously orderable graph G admits

- a spanning tree T such that, for any two vertices x, y of G, $d_T(x, y) \leq d_G(x, y) + 3$ (i.e., an *additive tree 3-spanner*) and
- a system $\mathcal{T}(G)$ of at most $O(\log n)$ spanning trees such that, for any two vertices x, y of G, a spanning tree $T \in \mathcal{T}(G)$ exists with $d_T(x, y) \leq d_G(x, y) + 2$ (i.e, a *system of at most* $O(\log n)$ *collective additive tree* 2*-spanners*).

These results generalize known results on tree spanners of dually chordal graphs and of distance-hereditary graphs. The results above are also complemented with some lower bounds which say that on some n-vertex homogeneously orderable graphs any system of collective additive tree 1-spanners must have at least $\Omega(n)$ spanning trees and there is no system of collective additive tree 2-spanners with constant number of trees.

1 Introduction

A spanning tree T of a graph G is called a *tree spanner* of G if T provides a "good" approximation of the distances in G. More formally, for $r \geq 0$, T is called an *additive tree* r*-spanner* of G if for any pair of vertices u and v their distance in T is at most r plus their distance in G (see [17,19]). A similar definition can be given for multiplicative tree t-spanners (see [6]); however in this paper we are only concerned with additive spanners. Tree spanners have many applications in various areas. They occur in biology and they can be used in message routing and as models for broadcast operations in communication networks. Tree spanners are useful also from the algorithmic point of view - many algorithmic problems are easily solvable on trees. If one needs to solve an NP-hard optimization problem concerning distances in a graph G and G admits a good tree spanner

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T, then an efficient solution to that problem on T would provide an approximate solution to the original problem on G.

The problem to decide for a graph G whether G has a multiplicative tree t–spanner (the *multiplicative tree t–spanner problem*) is NP–complete for any fixed $t \geq 4$ [6], and it remains NP-complete even on some rather restricted graph families. Fortunately, there is also a number of special classes of graphs where additive or multiplicative variants of the tree spanner problem are polynomial time solvable. Here we will mention only the results on two families of graphs which are relevant to our paper. Every dually chordal graph admits an additive tree 3-spanner [2] and every distance-hereditary graph admits an additive tree 2-spanner [19], and such tree spanners can be constructed in linear time.

In [12] we generalized the notion of tree spanners by defining a new notion of *collective tree spanners*. We say that a graph G *admits a system of μ collective additive tree r-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices u, v of G a spanning tree $T \in \mathcal{T}(G)$ exists such that the distance in T between u and v is at most r plus their distance in G. We say that system $\mathcal{T}(G)$ *collectively r-spans* the graph G and r is the *(collective) additive stretch factor*. Clearly, if G admits a system of μ collective additive tree r-spanners, then G admits an additive r-spanner with at most $\mu \times (n-1)$ edges, and if $\mu = 1$ then G admits an additive tree r-spanner. Note that, an induced cycle of length k provides an example of a graph which does not have any additive tree $(k-3)$-spanner, but admits a system of two collective additive tree 0-spanners. Furthermore, for any $r \geq 1$ there is a chordal graph which does not have any additive tree r-spanner [19]; on the other hand, any n-vertex chordal graph admits a system of $O(\log n)$ collective additive tree 2-spanners [12]. These two examples demonstrate the power of this new concept of collective tree spanners. One of the motivations to introduce this new concept stems from the problem of designing compact and efficient routing schemes in graphs. In [14,20], a shortest path routing labeling scheme for trees is described that assigns each vertex of an n-vertex tree a $O(\log^2 n / \log\log n)$-bit label. Given the label of a source vertex and the label of a destination, it is possible to compute in constant time, based solely on these two labels, the neighbor of the source that heads in the direction of the destination. Clearly, if an n-vertex graph G admits a system of μ collective additive tree r-spanners, then G admits a routing labeling scheme of deviation (i.e., additive stretch) r with addresses and routing tables of size $O(\mu \log^2 n / \log\log n)$ bits per vertex. Once computed by the sender in μ time, headers of messages never change, and the routing decision is made in constant time per vertex (for details see [11,12]). Other motivations stem from the generic problems of efficient representation of the distances in "complicated" graphs by the tree distances and of algorithmic use of these representations [1,7,13]. Approximating graph distance d_G by a simpler distance (in particular, by tree–distance d_T) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis.

Previously, collective tree spanners of particular classes of graphs were considered in [8,10,11,12,16]. Paper [12] showed that any chordal graph or chordal bipartite graph admits a system of at most $\log_2 n$ collective additive tree 2–spanners. These results were complemented by lower bounds, which say that any system of collective additive tree 1–spanners must have $\Omega(\sqrt{n})$ spanning trees for some chordal graphs and $\Omega(n)$ spanning trees for some chordal bipartite graphs. Furthermore, it was shown that any c-chordal graph admits a system of at most $\log_2 n$ collective additive tree $(2\lfloor c/2 \rfloor)$–spanners and any circular-arc graph admits a system of two collective additive tree 2–spanners. Paper [11] showed that any AT-free graph (graph without asteroidal triples) admits a system of two collective additive tree 2-spanners and any graph having a dominating shortest path admits a system of two collective additive tree 3-spanners and a system of five collective additive tree 2-spanners. In paper [8], it was shown that no system of constant number of collective additive tree 1-spanners can exist for unit interval graphs, no system of constant number of collective additive tree r-spanners can exist for chordal graphs and $r \leq 3$, and no system of constant number of collective additive tree r-spanners can exist for weakly chordal graphs and any constant r. On the other hand, [8] proved that any interval graph of diameter D admits an easily constructible system of $2\log(D-1)+4$ collective additive tree 1-spanners.

Only papers [10,16] have investigated (so far) collective tree spanners in the *weighted graphs*. Paper [10] demonstrated that any weighted graph with tree-width at most $k-1$ admits a system of $k \log_2 n$ collective additive tree 0–spanners, any weighted graph with clique-width at most k admits a system of $k \log_{3/2} n$ collective additive tree $(2\mathsf{w})$–spanners, and any weighted graph with size of largest induced cycle at most c (i.e., a c-chordal graph) admits a system of $4\log_2 n$ collective additive tree $(2(\lfloor c/3 \rfloor + 1)\mathsf{w})$–spanners (here, w is the maximum edge weight in G). The latter result was refined for weighted weakly chordal graphs: any such graph admits a system of $4\log_2 n$ collective additive tree $(2\mathsf{w})$-spanners. In [16], it was shown that any n–vertex planar graph admits a system of $O(\sqrt{n})$ collective multiplicative tree 1-spanners (equivalently, additive tree 0-spanners) and a system of at most $2\log_{3/2} n$ collective multiplicative tree 3–spanners.

In this paper we study collective additive tree spanners in homogeneously orderable graphs. The class of homogeneously orderable graphs was introduced in [4] to generalize the dually chordal graphs and the distance-hereditary graphs and to show that the Steiner tree problem can still be solved in polynomial time on this more general class of graphs. The follow up to [4] paper [9] showed also that both the connected r-domination problem and the r-dominating clique problem are polynomial time solvable on homogeneously orderable graphs. Homogeneously orderable graphs (HOGs) is a large family of graphs which comprises a number of well-known graph classes including Dually Chordal graphs, House-Hole-Domino-Sun-free graphs (HHDS-free graphs), Distance-Hereditary graphs, Strongly Chordal graphs, Interval graphs and others (see Figure 1). In Section 3, we show that every homogeneously orderable graph admits an additive tree 3-spanner constructible in linear time. In Section 4, we demonstrate

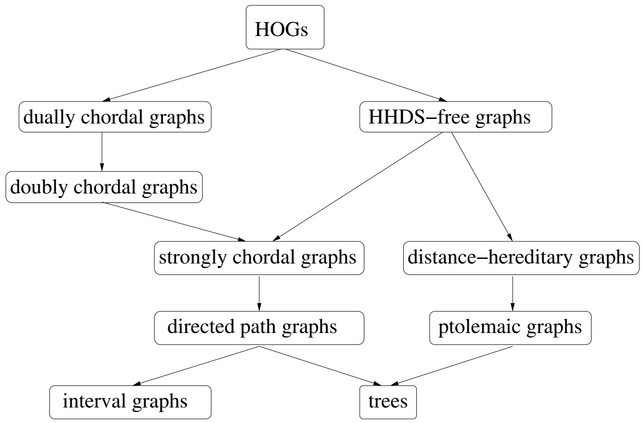

Fig. 1. Hierarchy of Homogeneously Orderable Graphs. For definitions of graph classes included see [5].

that every homogeneously orderable graph admits a system of $O(\log n)$ collective additive tree 2-spanners constructible in polynomial time. These results generalize known results on tree spanners of dually chordal graphs and of distance-hereditary graphs (see [2] and [19], respectively). Table 1 summarizes the results of this paper.

Table 1. Collective additive tree spanners of n-vertex homogeneously orderable graphs

additive stretch factor	upper bound on number of trees	lower bound on number of trees
3	1	1
2	$\log_2 n$	$c < \mu \leq \log_2 n$
1	$n-1$	$\Omega(n)$
0	$n-1$	$\Omega(n)$

2 Preliminaries

All graphs occurring in this paper are connected, finite, undirected, unweighted, loopless and without multiple edges. In a graph $G = (V, E)$ $(n = |V|, m = |E|)$ the *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ between the vertices u and v is the length of a shortest path connecting u and v. The i-th *neighborhood* of a vertex v of G is the set $N_i(v) = \{u \in V : d_G(v, u) = i\}$. For a vertex v of G, the sets $N(v) = N_1(v)$ and $N[v] = N(v) \cup \{v\}$ are called the *open neighborhood* and the *closed neighborhood* of v, respectively. For a set $S \subseteq V$, by $N[S] = \bigcup_{v \in S} N[v]$ we denote the *closed neighborhood* of S and by $N(S) = N[S] \setminus S$ the *open neighborhood* of S. The *disk* of radius k centered at v is the set of all vertices of distance at most k to v, i.e., $D_k(v) = \{u \in V : d_G(u, v) \leq k\} = \bigcup\{N_i(v) : i = 0, \ldots, k\}$.

Denote by $\mathcal{D}(G) = \{D_r(v) : v \in V,\ r$ a non-negative integer$\}$ the *family of all possible disks* of G and by $L(\mathcal{D}(G))$ *the intersection graph of those disks*, i.e., the vertices of $L(\mathcal{D}(G))$ are disks from $\mathcal{D}(G)$ and two vertices are adjacent if and only if the corresponding disks share a common vertex. Note that two disks $D_p(x)$ and $D_q(y)$ intersect if and only if $d_G(x, y) \leq p + q$. The *k–th power* G^k of a graph $G = (V, E)$ is the graph with vertex set V and edges between vertices u, v with distance $d_G(u, v) \leq k$. In what follows, a subset U of V is a *k–set* if U induces a clique in the power G^k, i.e., for any pair x, y of vertices of U we have $d_G(x, y) \leq k$. A graph G is called *chordal* if it does not have any induced cycle of length greater than 3.

We say that a graph $G = (V, E)$ *admits a system of* μ *collective additive tree r-spanners* if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that $d_T(x, y) \leq d_G(x, y) + r$. If $\mu = 1$, then the tree T such that $\mathcal{T}(G) = \{T\}$ is called a *tree r-spanner* of G.

A nonempty set $H \subseteq V$ is *homogeneous* in $G = (V, E)$ if all vertices of H have the same neighborhood in $V \setminus H$, i.e., $N(u) \cap (V \setminus H) = N(v) \cap (V \setminus H)$ for all $u, v \in H$, (any vertex $w \in V \setminus H$ is adjacent to either all or none of the vertices from H). A homogeneous set H is *proper* if $|H| < |V|$. Trivially for each $v \in V$ the singleton $\{v\}$ is a proper homogeneous set. Let U_1, U_2 be disjoint subsets of V. If every vertex of U_1 is adjacent to every vertex of U_2 then U_1 and U_2 form a *join*, denoted by $U_1 \bowtie U_2$. A set $U \subseteq V$ is *join–split* if U can be partitioned into two nonempty sets U_1, U_2 such that $U = U_1 \bowtie U_2$.

Next we recall the definition of homogeneously orderable graphs as given in [4]. A vertex v of $G = (V, E)$ with $|V| > 1$ is *h–extremal* if there is a proper subset $H \subset D_2(v)$ which is homogeneous in G and for which $D_2(v) \subseteq N[H]$ holds, i.e., H dominates $D_2(v)$. Thus, the sets H and $D_2(v) \setminus H$ form a join. A sequence $\sigma = (v_1, \ldots, v_n)$ is an *h–extremal ordering* of a graph G if for any $i = 1, \ldots, n-1$ the vertex v_i is h–extremal in $G_i := G(\{v_i, \ldots, v_n\})$ (induced subgraph of G formed by vertices of $\{v_i, \ldots, v_n\}$). A graph G is *homogeneously orderable* if G has an h–extremal ordering.

In [4] it is proved that a graph G is homogeneously orderable if and only if the square G^2 of G is chordal and each maximal two–set of G is join–split. This local structure of homogeneously orderable graphs implies a simple $O(n^3)$ recognition algorithm of this class, which uses the chordality of the squares (see [4]). Since the square G^2 of a graph G is chordal if and only if the graph $L(\mathcal{D}(G))$ is chordal [3], we can reformulate that characterization in the following way.

Theorem 1. [4] *A graph G is a homogeneously orderable graph if and only if the graph $L(\mathcal{D}(G))$ of G is chordal and each maximal two-set of G is join-split.*

This characterization will be very useful in Section 3 and Section 4. In Section 4, the following theorem from [9] will also be of use.

Theorem 2. [9] *For any homogeneously orderable graph $G = (V, E)$ with vertex function $r : V \to N$ and for any subset S of V, we have that S is r-dominated by some clique C of G (i.e, $d_G(v, C) \leq r(v)$ for every $v \in S$) if and only if $d_G(x, y) \leq r(x) + r(y) + 1$ for all $x, y \in S$.*

Finally, it is well known that the following lemma for chordal graphs holds.

Lemma 1 (Cycle Lemma for Chordal Graphs). *Let $C = (v_0, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{k-1})$ be a cycle of a chordal graph G with $k \geq 4$. Then, for any vertex v_i of C, $v_i v_j \in E(G)$, for $j \neq i-1, i, i+1 \pmod k$, or $v_{i-1}v_{i+1} \in E(G)$.*

3 Additive Tree 3-Spanners

In this section, we show that every homogeneously orderable graph G admits an additive tree 3-spanner. We prove that an algorithm similar to one presented in [2] for constructing an additive tree 3-spanner of a dually chordal graph constructs an additive tree 3-spanner of a homogeneously orderable graph. However, its correctness proof (Lemma 3 and Lemma 4) for homogeneously orderable graphs is more involved.

Assume u is an arbitrary vertex of a homogeneously orderable graph G and $k \in \{1, 2, \ldots, ecc(u)\}$, where $ecc(u) = \max\{d_G(u, v) : v \in V(G)\}$. Let $F_1^k, \ldots, F_{p_k}^k$ be the connected components of the graph obtained from G by removing vertices of $D_{k-1}(u)$. Denote $S_i^k = F_i^k \cap N_k(u)$ and $M_i^{k-1} = N(S_i^k) \cap N_{k-1}(u)$. M_i^{k-1} is called the *projection* of S_i^k to layer $N_{k-1}(u)$. Clearly, any two vertices $x, y \in S_i^k$ are connected by a path outside the disk $D_{k-1}(u)$. Denote by H_{k-1} the graph with $\bigcup_{i=1}^{p_k} M_i^{k-1}$ as the vertex set and two vertices x, y are adjacent in H_{k-1} if and only if they belong to a common set M_i^{k-1}.

Lemma 2. *Every connected component A of the graph H_k is a two-set in G.*

Proof. Let x, y be two vertices of a connected component A of the graph H_k. Then, we can find a collection of projections $M_{i_1}^k, M_{i_2}^k, \ldots, M_{i_h}^k$ such that $x \in M_{i_1}^k$, $y \in M_{i_h}^k$ and $M_{i_j}^k \cap M_{i_{j+1}}^k \neq \emptyset$ for all $j = 1, \ldots, h-1$. Pick $z_j \in M_{i_j}^k \cap M_{i_{j+1}}^k$, $j = 1, \ldots, h-1$ and let $z_0 := x$ and $z_h := y$. Since $z_{j-1}, z_j \in M_{i_j}^k$, $j = 1, \ldots, h$, we can find two vertices $v'_j, v''_j \in S_{i_j}^{k+1}$ adjacent to z_{j-1} and z_j, respectively. Let P_j be a path of $F_{i_j}^{k+1}$ connecting the vertices v'_j and v''_j. The disk $D_{k-1}(u)$ together with $D_1(x), D_1(y)$ and the disks of the family $\{D_1(z) : z \in \cup_{j=1}^h P_j\}$ forms a cycle in the intersection graph $L(\mathcal{D}(G))$. From chordality of this graph (see Theorem 1) and since $D_{k-1}(u) \cap D_1(z) = \emptyset$ holds for all $z \in \cup_{j=1}^h P_j$, we deduce (see the Cycle Lemma for Chordal Graphs) that $D_1(x) \cap D_1(y) \neq \emptyset$, i.e. $d_G(x, y) \leq 2$. □

Next, we are going to show that for every connected component A of the graph H_k there is a vertex $z \in N_{k-1}(u)$ such that $A \subseteq N(z)$. To show that, the following lemma is needed, proof of which is omitted in this version.

Lemma 3. *If $k \geq 2$ and A is a connected component of the graph H_k, then there is a vertex $t \in N_{k-2}(u)$ such that $A \subseteq N_2(t)$.*

Now, we are ready to prove the following lemma.

Lemma 4. *For every connected component A of the graph H_k there is a vertex $z \in N_{k-1}(u)$ such that $A \subseteq N(z)$.*

Proof. If $k = 1$, then the lemma clearly holds. Hence, assume $k \geq 2$. According to Lemma 3, there is a vertex $t \in N_{k-2}(u)$ such that $A \subseteq N_2(t)$. Hence, $A \cup \{t\}$ is a two-set of G. Let U be a maximal two-set of G such that $A \subseteq U$. By Theorem 1, U consists of two subsets U_1 and U_2 and $U_1 \bowtie U_2$. Clearly, $A \cup \{t\}$ must be in either U_1 or U_2, since $d_G(x,t) = 2$ for each $x \in A$. Without loss of generality, assume $A \cup \{t\}$ is in U_1. Then, U_2 must contain a vertex z which is adjacent to all the vertices in $A \cup \{t\}$. This vertex z can only be in $N_{k-1}(u)$. This concludes our proof. □

From the above lemmata, one can give the following linear time algorithm to construct an additive tree 3-spanner for a homogeneously orderable graph G.

PROCEDURE 1. Construct an additive tree 3-spanner for a HOG G

Input: A homogeneously orderable graph $G = (V, E)$.
Output: An additive tree 3-spanner T of G.
Method:

```
set E' = ∅;
pick an arbitrary vertex u in G;
set i = ecc(u);
for each vertex x ∈ N_i(u) do
   arbitrarily choose a vertex y ∈ N_{i−1}(u) such that xy ∈ E(G);
   add xy into E';
set i to i − 1;
while i ≥ 1 do
   construct the graph H_i and find its connected components;
   for each connected component A of H_i do
      find a vertex z ∈ N_{i−1}(u) such that A ⊆ N(z);
      for each vertex x ∈ A add xz into E';
   for other vertices x which are in N_i(u) but not in H_i;
      arbitrarily choose a vertex y ∈ N_{i−1}(u) such that xy ∈ E(G);
      add xy into E';
output T = (V, E').
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It is not hard to show that Procedure 1 can be implemented to run in linear time. The most complex step is the construction of the connected components of the graphs H_i ($i = ecc(u) - 1, \ldots, 2, 1$). We start from the layer $N_k(u)$, where $k = ecc(u)$, find its connected components $F_1^k, \ldots, F_{p_k}^k$ and contract each of them into a vertex. Then find the connected components $F_1^{k-1}, \ldots, F_{p_{k-1}}^{k-1}$ in the graph induced by $N_{k-1}(u)$ and the set of contracted vertices. To find the connected components of the graph H_{k-1}, we construct a special bipartite graph $B_{k-1} = (W, K; U)$. In this graph $W = \{f_1, \ldots, f_{p_k}\}$ (a vertex f_j represents component F_j^k), and K is the vertex set of H_{k-1} (which is $\bigcup_{i=1}^{p_k} M_i^{k-1} = \bigcup_{i=1}^{p_k}(N(F_i^k) \cap N_{k-1}(u))$. A vertex $f_j \in W$ and a vertex $v \in K$ are adjacent in B_{k-1} if and only if $v \in N(F_j^k)$. The graph B_{k-1} can be constructed in $O(\sum_{v \in N_k(u) \cup N_{k-1}(u)} deg(v))$ time. Note that a vertex $v \in N_{k-1}(u)$ belongs to H_{k-1} if and only if it has a neighbor in $N_k(u)$. The connected components of H_{k-1} are exactly the intersections of the connected components of B_{k-1} with the set K. After performing for H_{k-1} all operations prescribed in the other lines of Procedure 1, we contract each of $F_1^{k-1}, \ldots, F_{p_{k-1}}^{k-1}$ into a vertex and descend to the lower level. We repeat the above until we come to the vertex u.

Theorem 3. *The spanning tree T constructed by Procedure 1 is an additive tree 3-spanner of G.*

Proof. Let x, y be two arbitrary vertices of G. Assume $x \in N_{l_x}(u)$ and $y \in N_{l_y}(u)$. Without loss of generality, assume $l_x \leq l_y$. Let $P = (x_0 = x, x_1, \cdots, x_p = y)$ be a shortest path between x and y in G. Let k be the smallest integer such that $P \cap N_k(u) \neq \emptyset$. There are two cases to consider.

CASE 1: *There is exactly one vertex x_i in $P \cap N_k(u)$.*

First, consider the subcase when $i = 0$, i.e., $x_i = x_0 = x$. Let P_T be the path between y and u in T. Let y' be the vertex of P_T from $N_k(u)$. Clearly, x and y' belong to the projection M_j^k of that connected component F_j^{k+1} (of the induced subgraph of G formed by vertices $V \setminus D_k(u)$) which contains vertex y. By the way T was constructed, $d_T(x, y') \leq 2$ and $d_T(y, y') = l_y - l_x$ must hold. This implies $d_T(x, y) \leq d_T(x, y') + d_T(y', y) \leq 2 + l_y - l_x \leq 2 + d_G(x, y)$ since $d_G(x, y) \geq l_y - l_x$.

Let now $i \neq 0$. Since x_i is the only vertex in $P \cap N_k(u)$, x_{i-1} and x_{i+1} must be in $N_{k+1}(u)$. Let P'_T be the path between x and u and P_T be the path between y and u in T. Let x' and y' be the vertices of P'_T and P_T taken from $N_k(u)$. Clearly, x' and x_i belong to the projection $M_{j_x}^k$ of that connected component $F_{j_x}^{k+1}$ (of the induced subgraph of G formed by vertices $V \setminus D_k(u)$) which contains vertex x, and y' and x_i belong to the projection $M_{j_y}^k$ of that connected component $F_{j_y}^{k+1}$ which contains vertex y (it is possible that $F_{j_y}^{k+1} = F_{j_x}^{k+1}$). Since these projections $M_{j_y}^k$ and $M_{j_x}^k$ share a common vertex x_i, they must belong to the same connected component of the graph H_k. By the way T was constructed, $d_T(x', y') \leq 2$ and $d_T(y, y') = l_y - k$, $d_T(x, x') = l_x - k$ must hold. This implies $d_T(x, y) \leq d_T(x, x') + d_T(x', y') + d_T(y', y) \leq l_x - k + 2 + l_y - k \leq d_G(x, y) + 2$ since $d_G(x, y) = d_G(x, x_i) + d_G(x_i, y) \geq l_x - k + l_y - k$.

CASE 2: *There are at least two vertices x_i, x_j in $P \cap N_k(u)$.*

In this case, $d_G(x, y) > l_x - k + l_y - k$ (i.e., $d_G(x, y) \geq l_x + l_y - 2k + 1$) must hold. Again, let P'_T be the path between x and u and P_T be the path between y and u in T. Let x' and y' be the vertices of P'_T and P_T, respectively, taken from $N_k(u)$. Clearly, x', y', x_i and x_j are in one connected component F_t^k of the induced subgraph of G formed by vertices $V \setminus D_{k-1}(u)$. By the way T was constructed, this implies $d_T(x', y') \leq 4$. Hence, $d_T(x, y) \leq d_T(x, x') + d_T(y, y') + d_T(x', y') \leq$

$l_y + l_x - 2k + 4 \leq d_G(x, y) + 3$. This proves the second case and concludes the proof of the theorem. □

Note that in [2] an example of a dually chordal graph (and hence of a homogeneously orderable graph) is presented which does not have any additive tree 2-spanner. Thus, the additive stretch factor 3 for the tree spanners presented in Theorem 3 is best possible if one wants to achieve it only with one tree.

4 Collective Additive Tree 2-Spanners

In this section, we show that every homogeneously orderable graph H admits a system of $O(\log n)$ collective additive tree 2-spanners. According to [4], if a graph H is homogeneously orderable, then $G = H^2$ is a chordal graph. Unfortunately, the method developed in [12] for constructing collective tree spanners in some hereditary classes of graphs (so-called (α, r)*–decomposable graphs*) cannot be directly applied to the class of homogeneously orderable graphs as this class is not hereditary (see [4]). We will work first on the square $G = H^2$ of a homogeneously orderable graph H and then will move down to the original graph H. To the best of our knowledge this is the first non-trivial result on collective tree spanners of a non-hereditary family of graphs.

The following theorem is known for chordal graphs.

Theorem 4. [15] *Every n-vertex chordal graph G contains a maximal clique S such that if the vertices in S are deleted from G, every connected component in the graph induced by any remaining vertices is of size at most $n/2$.*

A linear time algorithm for finding for a chordal graph G a separating clique S satisfying the condition of the theorem is also given in [15]. We will call S a *balanced separator* of G.

Using the above theorem, one can construct for any chordal graph G a *(rooted) balanced decomposition tree* $\mathcal{BT}(G)$ as follows. If G is a complete graph, then $\mathcal{BT}(G)$ is a one-node tree. Otherwise, find a balanced separator S in G, which exists according to Theorem 4. Let $G_1, G_2, \ldots, G_p$ be the connected components of the graph $G \setminus S$ obtained from G by removing vertices of S. For each graph G_i $(i = 1, \ldots, p)$, which is also a chordal graph, construct a balanced decomposition tree $\mathcal{BT}(G_i)$ recursively, and build $\mathcal{BT}(G)$ by taking S to be the root and connecting the root of each tree $\mathcal{BT}(G_i)$ as a child of S. Clearly, the nodes of $\mathcal{BT}(G)$ represent a partition of the vertex set V of G into *clusters* $S_1, S_2, \ldots, S_q$ which are cliques. For a node X of $\mathcal{BT}(G)$, denote by $G(\downarrow X)$ the (connected) subgraph of G induced by vertices $\bigcup\{Y : Y \text{ is a descendent of } X \text{ in } \mathcal{BT}(G)\}$ (here we assume that X is a descendent of itself).

Consider two arbitrary vertices x and y of a chordal graph G and let $S(x)$ and $S(y)$ be the nodes of $\mathcal{BT}(G)$ containing x and y, respectively. Let also $NCA_{\mathcal{BT}(G)}(S(x), S(y)) = X_t$ be the nearest common ancestor and $X_0, X_1, \ldots,$ X_t be the sequence of common ancestors of $S(x)$ and $S(y)$ in $\mathcal{BT}(G)$ starting from the root X_0 of $\mathcal{BT}(G)$. We will need the following lemma from [12].

Lemma 5. [12] *Any path $P^G_{x,y}$, connecting vertices x and y in G, contains a vertex from $X_0 \cup \cdots \cup X_t$.*

Let now $H = (V, E)$ be a homogeneously orderable graph. By Theorem 1, $G = H^2$ is a chordal graph. We construct a *rooted balanced decomposition tree* $\mathcal{BT}(G)$ for G as described above. It is also known that any maximal clique of G is a join-split two–set of H. For any maximal clique C of G, let $C = U_1 \cup U_2$ such that, in H, $U_1 \bowtie U_2$. Note that X_i $(i = 0, 1, \ldots, t)$ may not be a maximal clique in G. Let C'_i be a maximal clique of G such that $X_i \subseteq C'_i$. Assume $C'_i = U^i_1 \cup U^i_2$ with $U^i_1 \bowtie U^i_2$ in H.

There are two cases to consider:

(a) $U^i_1 \cap X_i \neq \emptyset$ and $U^i_2 \cap X_i \neq \emptyset$;
(b) either $U^i_1 \cap X_i = \emptyset$ or $U^i_2 \cap X_i = \emptyset$ (without loss of generality, we assume $U^i_1 \cap X_i = \emptyset$ in this case).

For each connected graph $G(\downarrow X_i)$, let $H(\downarrow X_i)$ be the induced subgraph of H which has the same vertex set as $G(\downarrow X_i)$. Note that $H(\downarrow X_i)$ may not be connected as not all edges of $G(\downarrow X_i)$ are present in $H(\downarrow X_i)$. We construct a tree T_i for $H(\downarrow X_i)$ in the following way.

If $U^i_1 \cap X_i \neq \emptyset$ and $U^i_2 \cap X_i \neq \emptyset$ holds, then we choose two vertices $r_1 \in X_i \cap U^i_1, r_2 \in X_i \cap U^i_2$ and add r_1r_2 into $E(T_i)$. For each $x \in X_i \cap U^i_1$, we add xr_2 into $E(T_i)$. For each $y \in X_i \cap U^i_2$, we add yr_1 into $E(T_i)$. Then we extend the tree constructed so far by building a breadth-first-search-tree in $H(\downarrow X_i)$ starting at set X_i and spanning that connected component of $H(\downarrow X_i)$ which contains r_1 and r_2.

If $U^i_1 \cap X_i = \emptyset$ holds, then we choose a vertex r from U^i_1 (which may not be even in $H(\downarrow X_i)$) and construct a tree T_i as follows. For every vertex $x \in X_i$, we put xr into $E(T_i)$. Then, we extend the tree constructed so far by building a breadth-first-search-tree in $H(\downarrow X_i)$ starting at set X_i and spanning the vertices that can reach r via vertices in $H(\downarrow X_i)$.

Below we will prove that for some vertices $x, y \in H(\downarrow X_i)$ the tree T_i will guarantee $d_{T_i}(x, y) \leq d_H(x, y) + 2$. Let $x \in V(H(\downarrow X_i))$ and $x' \in X_i$ be vertices such that $d_{T_i}(x, x') = d_{H(\downarrow X_i)}(x, X_i)$. Vertex x' is called the *projection* of x in T_i. We use $P_{T_i}(x)$ to denote the shortest path between x' and x in T_i.

Consider any two vertices $x, y \in V$. Let as before $X_0, X_1, \ldots, X_t$ be the common ancestors of $S(x)$ and $S(y)$ in $\mathcal{BT}(G)$, and let X_i be the common ancestor with minimum i such that there is a shortest path $P^H_{x,y}$ between x and y in H with $P^H_{x,y} \cap X_i \neq \emptyset$. It is easy to see that such X_i must exist. Indeed, according to Lemma 5, any path $P^G_{x,y}$ between x and y in G must intersect $X_0 \cup \cdots \cup X_t$ and any path $P^H_{x,y}$ is also a path between x and y in G (as $G = H^2$ and therefore the edge set of H is a subset of the edge set of G). By the choice of X_i, any shortest path in H connecting x and y must be in $H(\downarrow X_i)$. Let T_i be the tree constructed as above for $H(\downarrow X_i)$. The following lemma can be proved to be true.

Lemma 6. $d_{T_i}(x, y) \leq d_H(x, y) + 2$ *must hold.*

Proof. Let $P^H_{x,y}$ be a shortest path of H connecting x and y. By the choice of X_i, this path is entirely in $H(\downarrow X_i)$ and intersects X_i. Let z_x, z_y be vertices of this path and X_i closest to x and y, respectively.

Let x', y' be the projections of x and y in T_i. According to the way T_i was constructed, $d_{T_i}(x', y') \le 3$, $d_{T_i}(x, x') = d_{H(\downarrow X_i)}(x, x') \le d_{H(\downarrow X_i)}(x, z_x) = d_H(x, z_x)$ and $d_{T_i}(y, y') = d_{H(\downarrow X_i)}(y, y') \le d_{H(\downarrow X_i)}(y, z_y) = d_H(y, z_y)$. If $d_{T_i}(x', y') \le 2$ or $z_x \ne z_y$, then we are done since $d_{T_i}(x, y) \le d_{T_i}(x, x') + d_{T_i}(x', y') + d_{T_i}(y, y') \le d_H(x, z_x) + d_{T_i}(x', y') + d_H(y, z_y) = d_H(x, y) - d_H(z_x, z_y) + d_{T_i}(x', y') \le d_H(x, y) + 2$. Hence, assume that $d_{T_i}(x', y') = 3$ and $d_H(x, y) = d_H(x, z_x) + d_H(y, z_y)$, i.e., $z_x = z_y$. Denote $s := z_x = z_y$, $d_H(x, s) := l_x$ and $d_H(y, s) := l_y$. Since $d_{T_i}(x', y') = 3$, by the way T_i was constructed, we have $U^i_1 \cap X_i \ne \emptyset$ and $U^i_2 \cap X_i \ne \emptyset$. Furthermore, vertices x' and y' can not be both in $U^i_1 \cap X_i$ or both in $U^i_2 \cap X_i$ (otherwise, $d_{T_i}(x', y') \le 2$). Without loss of generality, assume $x' \in U^i_1 \cap X_i$, $y' \in U^i_2 \cap X_i$ and $s \in U^i_1 \cap X_i$. Consider the following radius function for vertices s, y, y' in H: $r(s) = 0, r(y') = 0$ and $r(y) = l_y - 1$. According to Theorem 2, there is a clique C in H such that C r-dominates the set $\{s, y', y\}$. Hence, there must exist a vertex $y'' \in (C \setminus X_i)$ such that $sy'', y'y'' \in E(H)$ and $d_H(y'', y) = l_y - 2$. This implies that y'' is on a shortest path between x and y in H and, by the choice of X_i, y'' is in $H(\downarrow X_i)$. Since $sy'', y'y'' \in E(H)$, we have $d_H(y'', z) \le 2$ for any vertex $z \in U^i_1 \cup U^i_2$. Therefore, $\{y''\} \cup X_i$ is a clique in $G(\downarrow X_i)$. The latter contradicts with the fact that X_i was a maximal clique in $G(\downarrow X_i)$ (see Theorem 4 and paragraph on the construction of $\mathcal{BT}(G)$ after that theorem). Thus, the case that $d_{T_i}(x', y') = 3$ and $z_x = z_y$ is impossible. This concludes our proof. □

Let now $B^i_1, \ldots, B^i_{p_i}$ be the nodes on depth i of the tree $\mathcal{BT}(G)$. For each subgraph $H^i_j = H(\downarrow B^i_j)$ of H ($i = 0, 1, \ldots, depth(\mathcal{BT}(G)), j = 1, 2, \ldots, p_i$), denote by T^i_j the tree constructed for H^i_j as above. Clearly, T^i_j can be constructed in linear time. We call T^i_j a *local tree of* H. Since any two local trees T^i_j and $T^i_{j'}$ share at most one vertex, $T^i = \bigcup\{T^i_j, j = 0, \ldots, p_i\}$ is a forest. T^i can be extended to T'^i to span all the vertices in H. Tree T'^i is called a *global spanning tree of* H. By Lemma 6, we immediately get the following result.

Lemma 7. *For any two vertices $x, y \in V$, there exists a global spanning tree T'^i such that $d_{T'^i}(x, y) \le d_H(x, y) + 2$.*

By the way $\mathcal{BT}(G)$ was constructed, the depth of $\mathcal{BT}(G)$ is at most $\log_2 n$. Hence, there are at most $\log_2 n$ global spanning trees of H. Obviously, $G = H^2$ can be obtained in $O(nm)$ time. According to [12], $\mathcal{BT}(G)$ can be constructed in $O(n^2 \log_2 n)$ time (note that G may have $O(n^2)$ edges). Each T'^i is constructible in linear time. Therefore, the following theorem is true.

Theorem 5. *Any n-vertex homogeneously orderable graph admits a system of $\log_2 n$ collective additive tree 2-spanners which can be constructed in $O(nm + n^2 \log_2 n)$ time.*

5 Conclusion

In this paper we studied collective additive tree spanners in homogeneously orderable graphs. We showed that every n-vertex homogeneously orderable graph admits an additive tree 3-spanner constructible in linear time and a system of $O(\log n)$ collective additive tree 2-spanners constructible in polynomial time. These results generalize known results on tree spanners of dually chordal graphs and of distance-hereditary graphs. We mentioned also that there are homogeneously orderable graphs which do not admit any additive tree 2-spanners. Hence, it is natural to ask how many spanning trees are necessary to achieve collective additive stretch factor of 2 in homogeneously orderable graphs. We know that this number is not a constant (a proof of this is presented in the full version of the paper) and is not larger than $\log_2 n$. One may ask also how many spanning trees are necessary and how many are sufficient to achieve collective additive stretch factor of 1 or 0 in homogeneously orderable graphs. The answer to this question is simple. Any graph G on n vertices admits a system of at most $n-1$ collective additive tree 0-spanners. On the other hand, it is easy to see that any system of collective additive tree 1–spanners of homogeneously orderable graphs will need to have at least $\Omega(n)$ spanning trees (a proof of this is presented in the full version).

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