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On linear and circular structure of (claw, net)-free graphs

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Abstract

We prove that every (claw, net)-free graph contains an induced doubly dominating cycle or a dominating pair. Moreover, using LexBFS we present a linear time algorithm which, for a given (claw, net)-free graph, finds either a dominating pair or an induced doubly dominating cycle. We show also how one can use structural properties of (claw, net)-free graphs to solve efficiently the domination, independent domination, and independent set problems on these graphs.

Keywords: (Claw, net)-free graphs; Linear structure; Circular structure; Dominating cycle; Independent domination; Stability number; Linear-time algorithm

1. Introduction

Various classical families of graphs, such as interval graphs [25], permutation graphs [18], trapezoid graphs [9,14], and cocomparability graphs [20], enjoy a linear structure, which was usually described in terms of ad hoc properties of each of these classes of graphs. For example, in the case of interval graphs, the linear structure is traditionally expressed in terms of a linear order on the set of maximal cliques [3,4]. For permutation graphs the linear behavior is explained in terms of the underlying partial order of dimension two [2], for cocomparability graphs. The linear behavior is expressed in terms of the well-known linear structure of comparability graphs [24], and so on.

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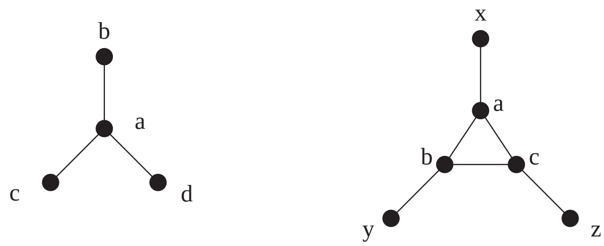

Fig. 1. The claw $K(a;b,c,d)$ and the net $N(a,b,c;x,y,z)$.

All these linearity properties are algorithmically useful for the recognition as well as for resolving a number of optimization problems on the classes mentioned above.

It turns out that each of these classes is a subfamily of a class of graphs called the asteroidal triple-free graphs (AT-free graphs, for short). An independent set of three vertices is called an asteroidal triple if between any pair in the triple there exists a path that avoids the neighborhood of the third. AT-free graphs were introduced by Lekkerkerker and Boland [25] who showed that a graph is an interval graph if and only if it is chordal and AT-free. Recently, Corneil, Olariu and Stewart have studied AT-free graphs with the stated goal of identifying the "agent" responsible for the linear behavior observed in the four subfamilies. In [10–12], they presented evidence that the property of being AT-free is what is enforcing the linear behavior of these classes.

One interesting "certificate" of linearity is the existence of a path (induced or shortest) such that every vertex outside the path is adjacent to some vertex on the path. Such a path is called an induced dominating path (or, respectively, dominating shortest path). It is shown in [10,11] that every connected AT-free graph contains an induced dominating path as well as a dominating shortest path, and such paths can be found in linear time. A stronger "certificate" of linearity of AT-free graphs was presented in [11,12]: every connected AT-free graph contains a dominating pair of vertices, that is, a pair of vertices with the property that every path connecting them is a dominating path. In [12], based on the well-known Lexicographic Breadth-First Search (LexBFS) of [27], the authors gave a simple, linear time algorithm for computing a dominating pair in a connected AT-free graph.

In this paper, we investigate structure properties and algorithmic implications of the (claw, net)-free graphs (CN-free graphs, for short), i.e. the graphs containing no induced claw and no induced net (see Fig. 1). These graphs turn out to be closely related to AT-free graphs from their structure properties but are incomparable with them since AT-free graphs are in general not claw-free and the net is giving just one example of an asteroidal triple. Note that CN-free graphs may contain arbitrarily long induced cycles, whereas AT-free graphs by definition contain no induced cycles of length at least six.

CN-free graphs are well known for their Hamiltonicity properties [17] as is described below; they are also known in the literature by the fact that the so-called struction works well for solving the maximum independent set problem on these graphs [21].

There were, however, only a few results on the structure of these graphs. The main contribution of this paper is to provide a number of results on the linear and circular structure of CN-free graphs which imply further algorithmic results. In particular, we show that every CN-free graph has a dominating pair or an induced doubly dominating cycle, i.e. an induced cycle such that every vertex outside the cycle has at least two neighbors on the cycle. Moreover, using LexBFS, we give a linear time algorithm which for a given CN-free graph finds either a dominating pair or an induced doubly dominating cycle. Note that not every CN-free graph has a dominating pair as the example of an induced cycle with more than six vertices shows. Thus, these graphs have a linear or a circular structure. The existence of an induced dominating cycle (or an induced doubly dominating cycle) in a graph may be considered as a "certificate" of the circular structure of the graph. Clearly, all proper circular arc graphs [30] are CN-free, and circular arc graphs [29] which are not interval graphs have a dominating cycle and, hence, enjoy such a kind of circular structure. Note that also AT-free graphs can be generalized in an obvious way in order to admit circular structure. Some of the algorithmic implications remain true for such graphs.

CN-free graphs were introduced by Duffus et al. [17] already in the 1980s. Although this class of graphs seems to be rather restrictive, it contains a couple of graph families, that are of interest in their own right. Examples of those families are unit interval graphs, claw-free AT-free graphs and proper circular arc graphs. In their paper [17] Duffus et al. showed that every connected CN-free graph contains a Hamiltonian path and every two-connected CN-free graph contains a Hamiltonian cycle. Later, Shepherd [28] proved that there is an $O(n^6)$ algorithm for finding such a Hamiltonian path/cycle in CN-free graphs. Note also that CN-free graphs are exactly the Hamiltonian-hereditary graphs [13], i.e. the graphs for which every connected induced subgraph contains a Hamiltonian path.

In [5] we gave a constructive existence proof and presented linear time algorithms for the Hamiltonian path and Hamiltonian cycle problems on CN-free graphs. The important structural property that we exploited for this, is the existence of an induced dominating path in every connected CN-free graph, and the existence of a good pair or an induced doubly dominating cycle in every two-connected CN-free graph. A *good pair* is a pair of vertices, such that there exist two internally disjoint induced dominating paths connecting these vertices. For a connected CN-free graph we presented a linear time algorithm, which finds an induced dominating path, and showed how one can use this path to construct a Hamiltonian path in linear time. For a two-connected CN-free graph we gave a linear time algorithm which finds either a good pair or an induced doubly dominating cycle. Again, given an induced doubly dominating cycle or a good pair of a CN-free graph, a Hamiltonian cycle can be constructed in linear time [5]. Note that, having a dominating pair instead of good pair will make the algorithm presented in [5] for the Hamiltonian cycle problem much simpler.

This paper is organized as follows. The remaining part of this section establishes notation and terminology that will be used throughout the paper. In Section 2 we prove the existence of an induced doubly dominating cycle or a dominating pair in every connected CN-free graph. In Section 3 we present a linear time algorithm which, for a given connected CN-free graph, finds either a dominating pair or an induced doubly dominating cycle. Section 4 shows how one can use structural properties of CN-free graphs to solve efficiently the domination, independent domination, and independent set problems on CN-free graphs.

For terms not defined here, we refer to [15,19]. For definitions and properties of special graph classes, we refer to [7,19]. In this paper we consider finite connected undirected graphs $G=(V,E)$ without loops and multiple edges. The cardinality of the vertex set is denoted by n, whereas the cardinality of the edge set is denoted by m.

A *path* is a sequence of vertices $(v_0,\dots,v_l)$ such that all v_i are distinct and $v_iv_{i+1}\in E$ for $i=0,\dots,l-1$; its *length* is l. An *induced path* is a path where $v_iv_j\in E$ iff $i=j-1$ and $j=1,\dots,l$. A *cycle* (*k-cycle*) is a path $(v_0,\dots,v_k)$ $(k\geqslant 3)$ such that $v_0=v_k$; its *length* is k. An *induced cycle* is a cycle where $v_iv_j\in E$ iff $|i-j|=1$ (modulo k). A *hole* H_k is an induced cycle of length $k\geqslant 5$.

The *distance* $dist(v,u)$ between vertices v and u is the smallest number of edges in a path joining v and u. The *eccentricity* $ecc(v)$ of a vertex v is the maximum distance from v to any vertex in G. The *diameter* $diam(G)$ of G is the maximum eccentricity of a vertex in G. A pair v,u of vertices of G with $dist(v,u)=diam(G)$ is called a *diametral pair*.

For every vertex we denote by $\mathrm{N}(v)$ the set of all neighbors of v, $\mathrm{N}(v)=\{u\in V\colon dist(u,v)=1\}$. The *closed neighborhood* of v is defined by $\mathrm{N}[v]=\mathrm{N}(v)\cup\{v\}$. For a vertex v and a set of vertices $S\subseteq V$, the minimum distance between v and vertices of S is denoted by $dist(v,S)$. The *closed neighborhood* $\mathrm{N}[S]$ of a set $S\subseteq V$ is defined by $\mathrm{N}[S]=\{v\in V\colon dist(v,S)\leqslant 1\}$. We say that a set $S\subseteq V$ *dominates* G if $\mathrm{N}[S]=V$, and S *doubly dominates* G if every vertex of $G\setminus S$ has at least two neighbors in S. Then, we say that S is a *dominating* (resp. *doubly dominating*) *set* of G. A *dominating pair* of G is a pair of vertices $v,u\in V$ such that every induced path between v and u dominates G.

The *claw* is the induced complete bipartite graph $K_{1,3}$ and for simplicity, we refer to it by $K(a;b,c,d)$ (see Fig. 1). The *net* is the induced six-vertex graph $N(a,b,c;x,y,z)$ shown in Fig. 1. A graph is called CN-free, or equivalently (*claw*, *net*)-*free* if it contains neither a claw nor a net. An *asteroidal triple* of G is a triple of pairwise non-adjacent vertices, such that for each pair of them there exists a path in G that does not contain any vertex in the (close) neighborhood of the third one. A graph is called AT-free if it does not contain an asteroidal triple. Finally, an *independent set* of G is a set of pairwise non-adjacent vertices of G.

2. Structural results

In this section we prove the existence of an induced doubly dominating cycle or a dominating pair in every CN-free graph. To prove the main theorem of this section we will need the following auxiliary results. Recall that we consider only connected graphs in this paper.

Lemma 2.1. *Every hole of a CN-free graph G doubly dominates G.*

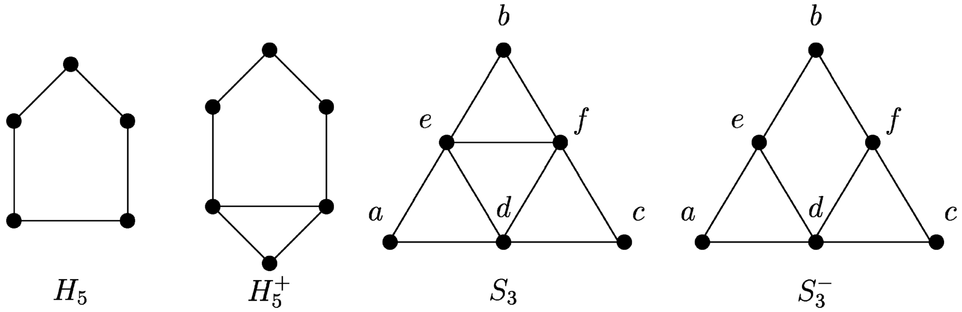

Fig. 2.

Proof. Let $H=(x_0,x_1,\ldots,x_{k-1},x_0)$ ($k \geqslant 5$) be a hole in G. If $V \setminus \mathrm{N}[H]$ is not empty (i.e., H does not dominate G), then there exists a vertex v such that $dist(v,H)=2$. Let x be a neighbor of v in $\mathrm{N}[H]$. If x is adjacent to only one vertex of H, say x_i, then we get a claw $K(x_i;x,x_{i-1},x_{i+1})$ (all additions here are modulo k). If x is adjacent to two nonadjacent vertices of H, say x_i,x_j, then we have a claw $K(x;v,x_i,x_j)$. Hence, we may assume that x is adjacent only to two adjacent vertices of H, say x_i,x_{i+1}. But now a net $N(x,x_i,x_{i+1};v,x_{i-1},x_{i+2})$ arises (note that since $k \geqslant 5$, vertices x_{i+2} and x_{i-1} are not adjacent). Consequently, H dominates G and, since G is claw-free, it is a doubly dominating cycle. □

A *subgraph* G' of G (*doubly*) *dominates* G if the vertex set of G' (doubly) dominates G.

Lemma 2.2. *Every induced subgraph of a CN-free graph G which is isomorphic to S_3 or S_3^- (see Fig. 2) doubly dominates G.*

Proof. Let G contain an induced subgraph isomorphic to S_3^-, and assume that it does not dominate G. Then, there must be a vertex s such that $dist(s,S_3^-)=2$. Let x be a neighbor of s from $\mathrm{N}[S_3^-]$. If x is adjacent neither to a, nor to b, nor to c (see Fig. 2), then G contains a claw (e.g., if $xf \in E$ then a claw $K(f;b,c,x)$ arises). Thus, without loss of generality, x has to be adjacent to a or b.

If $xa \in E$ then x is adjacent neither to b nor to c, since otherwise we will get a claw ($K(x;a,b,s)$ or $K(x;a,c,s)$). To avoid a net $N(a,e,d;x,b,c)$ vertex x must be adjacent to e or d. But, if $ex \in E$ then $xd \in E$ too (otherwise we will have a claw $K(e;b,d,x)$). Analogously, if $xd \in E$ then also $xe \in E$. Hence, x is adjacent to both e and d, and a net $N(x,e,d;s,b,c)$ arises.

Now, we may assume that x is adjacent to b and not to a,c. To avoid a claw $K(b;x,e,f)$, x must be adjacent to e or f. But again, $xe \in E$ if and only if $xf \in E$ (otherwise we get a net $N(x,b,e;s,f,a)$ or $N(x,b,f;s,e,c)$). Hence x is adjacent to both e and f and a claw $K(x;s,e,f)$ arises.

The contradictions obtained show that S_3^- dominates G. Moreover, it is easy to see that it doubly dominates G. Similarly, every S_3 (if it exists) doubly dominates G. □

Corollary 2.3. *A CN-free graph of diameter greater than* 3 *cannot contain the graphs* H_5, S_3, S_3^- *as induced subgraphs.*

Proof. Let G contain an induced subgraph isomorphic to S_3^- and v, u be a diametral pair of vertices of G, i.e. $dist(v,u) = diam(G)$. Since $dist(v,u) \geqslant 4$ and by Lemma 2.2, v and u do not belong to S_3^-, but are adjacent to at least two vertices of it. Now, it is easy to observe that in the uniquely possible position of v and u we will get a forbidden claw. Analogously, G cannot contain H_5, S_3 as an induced subgraph. □

Corollary 2.4. *Let H be a hole of a CN-free graph G of diameter greater than* 3. *Every vertex from $V \setminus H$ is adjacent exactly to two, three or four consecutive vertices of H.*

Proof. Let a vertex $x \in V \setminus H$ be adjacent to five or more vertices of a hole H. Since the length of H is at least 6, among these neighbors of x we will have three pairwise non-adjacent vertices, i.e. a claw with the center at x will arise. Furthermore, if x is adjacent to a vertex y of H then x is adjacent to a neighbor of y on H as well (otherwise, we will get a claw). From this and Lemma 2.1, we conclude that $p := |\mathrm{N}[x] \cap \mathrm{H}| \in \{2,3,4\}$, and if $p \in \{2,3\}$ then the neighbors of x on H form a path. Let now $p = 4$ and assume that neighbors of x on H do not form a path, i.e. $\mathrm{N}[x] \cap H = \{x_i, x_{i+1}, x_j, x_{j+1}\}$, where $j \neq i+2$, and $i \neq j+2$. Then evidently G contains a net $N(x_i, x_{i+1}, x; x_{i-1}, x_{i+2}, x_j)$ (if $j > i+3$) or a graph S_3^- with vertices $x_i, x_{i+1}, x, x_{i+2}, x_j, x_{j+1}$ as an induced subgraph (if $j = i+3$). By Corollary 2.3, this is impossible. □

The proof of the following important lemma can be found in [5].

Lemma 2.5. *Let P be an induced path connecting vertices v and u of a CN-free graph G. Let also s be a vertex of G such that $s \notin \mathrm{N}[P]$ and $dist(v,s) \leqslant dist(v,u)$. Then*

(1) *for every shortest path P' connecting v and s, $P' \cap P = \{v\}$ holds, and*
(2) *if there is an edge xy of G such that $x \in P \setminus \{v\}$ and $y \in P' \setminus \{v\}$, then both x and y are neighbors of v.*

A pair of vertices u, v of G with $dist(u,v) = ecc(u) = ecc(v)$ is called a *pair of mutually furthest vertices.* Evidently, if $dist(x,y) = diam(G)$ then x, y are mutually furthest vertices of G.

Lemma 2.6. *Let G be a CN-free graph, not containing the graphs S_3, S_3^-, H_5^+ (see Fig.* 2) *and H_k ($k \geqslant 6$) as induced subgraphs. Then, every pair of mutually furthest vertices forms a dominating pair of G.*

Proof. Let u, v be a pair of mutually furthest vertices of G and P be an induced path connecting v and u. Assume that $V \setminus \mathrm{N}[P]$ is not empty. Hence, we have $dist(v,u) \geqslant 2$.

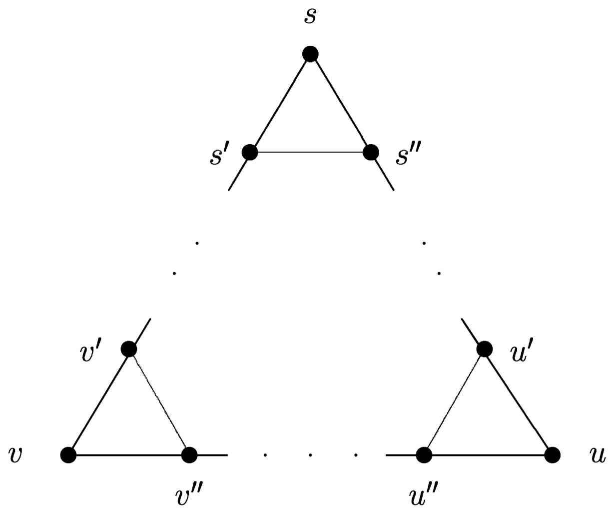

Fig. 3.

Consider a vertex $s \in V \setminus \mathrm{N}[P]$ and shortest paths P_v and P_u connecting vertex s with v and u, respectively. Both these paths have lengths at least 2.

Since u, v are mutually furthest vertices in G, $dist(v,s) \leqslant dist(v,u)$ and $dist(u,s) \leqslant dist(u,v)$. Hence, by Lemma 2.5, $P \cap P_v = \{v\}$ and $P \cap P_u = \{u\}$. Moreover, if there is a chord between P and P_v, then it is unique and both its endvertices are adjacent to v. A similar fact holds for P and P_u, both endvertices of the chord (if it exists) are adjacent to u.

Now, without loss of generality, we suppose that $dist(s,u) \leqslant dist(s,v)$. Then from $u \notin \mathrm{N}[P_v]$ and Lemma 2.5 we deduce that $P_u \cap P_v = \{s\}$ and between paths P_v and P_u at most one chord is possible, namely that one with both endvertices adjacent to s. Consequently, we have constructed an induced subgraph of G shown in Fig. 3 (only chords $s's''$, $v'v''$ and $u'u''$ are possible in the cycle).

It is easy to see that, if the lengths of all three paths P, P_v, P_u are at least 3, then G has a hole H_k ($k \geqslant 6$). Furthermore, if at least one of these paths has length greater than 4, or two of them have length 3, then G must contain a hole H_k ($k \geqslant 6$) or graph H_5^+ as an induced subgraph. It remains to consider two cases: the lengths of both P_v and P_u are 2 and the length of P is 3 or 2. In both these cases, clearly, graph G contains either a hole H_k ($k \in \{6,7\}$) or an induced subgraph isomorphic to H_5^+, or to S_3^-, or to S_3. □

Corollary 2.7. *Let G be a CN-free graph of diameter greater than* 3 *that does not contain any hole H_k ($k \geqslant 6$) as an induced subgraph. Then, every pair of mutually furthest vertices of G forms a dominating pair.*

Proof. It follows from Lemma 2.6 and Corollary 2.3. □

Theorem 2.8. *Every CN-free graph G has an induced doubly dominating cycle or a dominating pair.*

Proof. By Lemmas 2.1 and 2.6, we may assume that G does not contain holes H_k ($k \geqslant 5$) but has an induced subgraph isomorphic to S_3 or S_3^-.

Let G contain a S_3^- with vertex labeling shown in Fig. 2. We claim that the induced cycle (e,b,f,d,e) doubly dominates G. Indeed, if a vertex s of G does not belong to S_3^-, then, by Lemma 2.2, it is adjacent to two vertices of S_3^-. Suppose that s is adjacent to none of e,b,f,d. Then, $sa, sc \in E$ and we obtain a hole $H_6=(s,a,e,b,f,c,s)$ of G. Hence, (e,b,f,d,e) dominates G, and since G is claw-free this cycle is doubly dominating.

Let now G contain a S_3 with vertex labeling shown in Fig. 2. We will show that every vertex of G is adjacent to at least two vertices of the cycle (e,f,d,e). Suppose vertex s of G is adjacent to neither of e,d. Then, by Lemma 2.2, s is adjacent to at least two of a,b,c,f. Let $sf \in E$. To avoid a claw, vertex s is adjacent to both b and c. But then a hole $H_5=(s,b,e,d,c,s)$ arises. Therefore, $sf \notin E$ and, without loss of generality, we assume that $sa, sc \in E$. Then a hole $H_5=(s,a,e,f,c,s)$ occurs. □

3. Algorithm

In this section we present a linear time algorithm that, for a given CN-free graph G, finds either a dominating pair or an induced doubly dominating cycle.

In what follows we will use a special ordering of the vertex set of a graph G produced by the well-known *Lexicographic Breadth-First-Search* (*LexBFS*) of Rose et al. [27].

Recall that *LexBFS* is a refinement of the *Breadth-First-Search* and orders the vertices of a graph by assigning numbers from $n=|V|$ to 1 in the following way: assign the number k to a vertex v (as yet unnumbered) which has lexically largest vector (s_i: $i=n,n-1,\ldots,k+1$), where $s_i=1$ if v is adjacent to the vertex numbered i, and $s_i=0$ otherwise. An ordering $\sigma=v_1,v_2,\ldots,v_n$ of the vertex set of a graph G generated by LexBFS will be called *LexBFS-ordering* of G. A LexBFS-ordering of a graph G can be found in linear time [27].

We will often use the following property of LexBFS-orderings (cf. [23]):

(P1) If $a<b<c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c<d$, $db \in E$ and $da \notin E$.

We write $a<b$ whenever in a given ordering σ vertex a has a smaller number than vertex b. Moreover, $a<\{b,c\}$ is an abbreviation for $a<b$, $a<c$. It is well known that any LexBFS-ordering has property (P1) [19]. Moreover, any ordering fulfilling (P1) can be generated by LexBFS [6].

3.1. Finding a hole

Let G be a graph and σ be a LexBFS-ordering of G. For two vertices a and b of G, denote by b_a the vertex with the largest number in σ such that $ab_a \notin E$ but

$bb_a \in E$ (if such a vertex exists). For future needs we will show here that, for a given pair of vertices a and b, the vertex b_a can be found in time $deg(a)$ as follows. Let $(x_1,\dots,x_l)$ and $(y_1,\dots,y_k)$ be ordered adjacency lists of b and a, respectively, given in increasing order with respect to σ. Note that ordered adjacency lists of a graph G can be computed from unordered ones in linear time (see [19]). So, we may assume that our graph is given with the ordered adjacency lists. Now to find the vertex b_a in $deg(a)$ time we apply the following method.

```
b_a := x_l;
do while k ⩾ 1
   do case
      case y_k < x_l /* bx_l ∈ E, ax_l ∉ E */
         exit
      case y_k > x_l /* ay_k ∈ E, by_k ∉ E */
         k := k − 1
      case y_k = x_l /* ax_l, bx_l ∈ E */
         k := k − 1;
         l := l − 1;
         b_a := x_l
multicolumn3l      endcase
   enddo
```

We will say that vertices a,b,c of a graph G form *a promising triple* if $a<b<c$, $ab,ac \in E, bc \notin E$ and the vertex b_a is not adjacent to c. Note, that by property (P1) a vertex b_a with $bb_a \in E$, $ab_a \notin E$ and $c<b_a$ always exists.

Lemma 3.1 (Dragan [16]). *Let G be an arbitrary graph and σ be a LexBFS-ordering of G. Then every given promising triple of G can be extended to a hole in linear time.*

Proof. Let a,b,c be a promising triple of G, i.e. $a<b<c$, $ab,ac \in E$, $bc \notin E$ and the vertex b_a is not adjacent to c. For convenience denote by x_1,x_2,x_3 and x_4 vertices a,b,c and b_a, respectively.

We have $x_1<x_2<x_3<x_4$ and an induced path $P_4=(x_4,x_2,x_1,x_3)$. From the choice of $b_a=x_4$, we conclude that, for any vertex $y>x_4$, $yx_2 \in E$ if and only if $yx_1 \in E$. Note that if $yx_1 \in E$ but $yx_2 \notin E$, then by (P1) there must be a vertex $t>y>x_4$ adjacent to x_2 and not to x_1, which contradicts the maximality of x_4.

Now, to construct a hole in linear time we do the following. We find in $deg(x_2)$ time a vertex x_5 which is adjacent to x_3, not adjacent to x_2, and has the largest number in σ (see method described above). Since $x_2<x_3<x_4$, $x_2x_4 \in E$, $x_3x_4 \notin E$, by (P1), such a vertex exists and its number in σ is larger than the number of x_4. Clearly, $x_5x_1 \notin E$ since $x_5x_2 \notin E$ and $x_5>x_4$. Again, from the choice of x_5, for any vertex $y>x_5$, $yx_3 \in E$ if and only if $yx_2 \in E$ (if and only if $yx_1 \in E$). If vertices x_4 and x_5 are adjacent, then we have constructed a hole $H_5=(x_1,x_3,x_5,x_4,x_2,x_1)$. Otherwise, if $x_4x_5 \notin E$, then vertices $x_1<x_2<x_3<x_4<x_5$ form an induced path $P_5=(x_4,x_2,x_1,x_3,x_5)$ and we continue by

finding in $deg(x_3)$ time a new vertex x_6 which has the largest number in σ and fulfills $x_6x_4 \in E$, $x_6x_3 \notin E$. Then, we will have either a hole $H_6 = (x_1, x_3, x_5, x_6, x_4, x_2, x_1)$ or an induced path $P_6 = (x_6, x_4, x_2, x_1, x_3, x_5)$ formed by vertices $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ with the property that, for every $y > x_6$, $yx_i \in E$ if and only if $yx_{i-1} \in E$ $(2 \leqslant i \leqslant 4)$.

Since graph G is finite, at a certain step, the induced path $P_k = (x_k, x_{k-2}, \ldots, x_2, x_1, x_3, \ldots, x_{k-1})$ $(k \geqslant 6)$, formed by vertices $x_1 < x_2 < \ldots < x_{k-1} < x_k$ with the property that, for every $y > x_k$, $yx_i \in E$ if and only if $yx_{i-1} \in E$ $(2 \leqslant i \leqslant k-2)$, will be extended in $deg(x_{k-2})$ time by a new vertex x_{k+1} $(x_{k+1} > x_k,\ x_{k+1}x_{k-1} \in E,\ x_{k+1}x_{k-2} \notin E)$ to a hole $H_{k+1} = (x_1, x_3, \ldots, x_{k-1}, x_{k+1}, x_k, \ldots, x_2, x_1)$. From the construction, the time we spent is $\Sigma_{i=2}^{k-2} deg(x_i) = O(m)$. □

The *smallest* (*largest*) vertex of a set S is a vertex of S which has the smallest (largest) number with respect to σ.

Lemma 3.2. *Let G be a CN-free graph of diameter greater than* 3 *and σ be a LexBFS-ordering of G. Then in every hole H of G the smallest vertex together with its neighbors form a promising triple.*

Proof. Let H be a hole of G and a be the smallest vertex of H. Let also b and c be the neighbors of a on H and assume, without loss of generality, that $b < c$. If a, b, c is not a promising triple, then the vertex b_a is adjacent to c and hence does not belong to H. Thus, we get a vertex outside of H which is adjacent to b, c and not to a. Since the length of H is at least 6 (see Corollary 2.3), the neighbors of b_a on H do not form a path, contradicting Corollary 2.4. □

For a vertex $x \in V$, denote by $ln(x)$ the largest vertex of $N[x]$ and by $sn(x)$ the smallest vertex u of $N(x) \setminus N[ln(x)]$ such that $u > x$ (if such a vertex u does not exist, then $sn(x)$ is undefined). Note that $x = ln(x)$ if and only if x has the number n in σ.

Let now $F \subset V$ be the set of all $ln(\cdot)$ vertices from V, i.e. $v \in F$ if and only if there exists a vertex $u \in V$ such that $v = ln(u)$. Let also $S_v = \{u \in V\colon v = ln(u)$ and the vertex $sn(u)$ exists$\}$. We say that a promising triple a, b, c is *extreme* if $c \in F$, a is the largest vertex of S_c and $b = sn(a)$. See Fig. 4 for an illustration.

Lemma 3.3. *Let G be a CN-free graph of diameter greater than* 3 *and σ be a LexBFS-ordering of G. If G has a promising triple, then it has an extreme promising triple.*

Proof. The *sum of a promising triple a, b, c* with $a < b < c$ is the sum $\sigma(a) + \sigma(c)$ of the numbers of a and c in σ. Let a, b, c be a promising triple of G which has the largest sum $\Sigma = \sigma(a) + \sigma(c)$. Let also $H = (a, c, f, h, \ldots, g, d, b, a)$ be the hole generated by the method described in Lemma 3.1, i.e. $d = b_a$, $f = c_b$ and $a < b < c < d < f < g \leqslant h$ hold. By Corollary 2.3, each hole of G is of length at least 6 (so, $h{=}g$ is not excluded).

Assume that $c \neq v := ln(a)$. If v is adjacent neither to b nor to d, then vertices a, b, v form a promising triple with the sum $\sigma(a) + \sigma(v)$ larger than Σ. Furthermore, if $vd \in E$ then, by Corollary 2.4, $vb \in E$ as well. Hence, in any case, vertices v and b

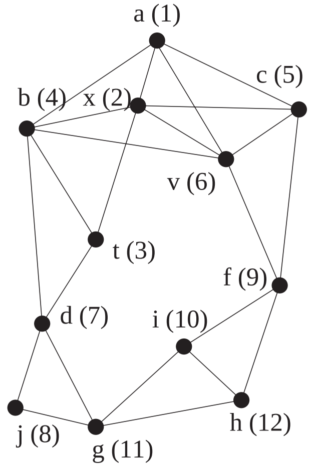

Fig. 4. A CN-free graph with a LexBFS-ordering a,x,t,b,c,v,d,j,f,i,g,h. a,b,c is a promising triple since $a<b<c$, $ab,ac\in E$, $bc,b_ac\notin E$ where $b_a=d$. This promising triple can be extended to a hole (a,b,d,g,h,f,c,a). $ln(a)=ln(x)=v$, $sn(x)=t$, $sn(a)$ is undefined since $N(a)\setminus N[v]=\emptyset$. $F=\{h,g,f,d,v\}$, $S_v=\{x\}$, $S_d=\emptyset$. x,t,v is an extreme promising triple since $x<t<v$, $xt,xv\in E$, $tv,t_xv\notin E$ (where $t_x=d$) and $v\in F$, x is the only vertex of S_v and $t=sn(x)$.

have to be adjacent. If v is adjacent also to f, then $\mathrm{N}[v]\cap H=\{a,b,c,f\}$ by Corollary 2.4, and a new hole $H'=H\setminus\{a,c\}\cup\{v\}$ with the smallest vertex b gives, by Lemma 3.2, a new promising triple b,v,d (or b,d,v) with the sum larger than Σ. Analogously, if v is adjacent to c and $vf\notin E$, then either $\mathrm{N}[v]\cap H=\{a,b,c\}$ and we have a new hole $H'=H\setminus\{a\}\cup\{v\}$ with the smallest vertex b, or $\mathrm{N}[v]\cap H=\{a,b,c,d\}$ and we have a new hole $H'=H\setminus\{a,b\}\cup\{v\}$ with the smallest vertex c. In both cases we get a promising triple with the sum larger than Σ. Thus, we conclude that $bv\in E$ but $cv,fv\notin E$. Since a,c,v also cannot be a promising triple, vertex c_a must be adjacent to v. Recall that c_a is the vertex which has the largest number in σ and fulfills $cc_a\in E$, $ac_a\notin E$. From the choice of c_a, $c_a>f$ holds. Then the choice of the vertex $f=c_b$ gives $bc_a\in E$, and a contradiction with Corollary 2.4 arises.

Thus, $c=ln(a)$. Assume now that there is a vertex $x\in S_c$ such that $x>a$, and let $t=sn(x)$. We have $c=ln(x)$, $tx\in E$, $tc\notin E$ and $c>t>x$. Since $f>d>c=ln(x)$, $xd,xf\notin E$ (otherwise, c cannot be $ln(x)$). Also $b>x$ holds because $ln(b)\geqslant d>c=ln(x)$ and σ is a LexBFS-ordering of G. To avoid a claw $K(c;f,a,x)$, vertices a and x are adjacent. Furthermore, $xb\notin E$, since otherwise x,b,c would give a promising triple with the sum $\sigma(x)+\sigma(c)$ larger than Σ (x is the smallest vertex of the hole $H'=H\setminus\{a\}\cup\{x\}$). Thus, by Corollary 2.4, $\mathrm{N}[x]\cap H=\{a,c\}$. If $tb\in E$ then in the cycle $(x,c,f,h,\ldots,g,d,b,t,x)$ only the chord td is possible (if t is adjacent to a vertex $y\in H\setminus\{a,b,d,c\}$, then a claw $K(t;x,b,y)$ will arise). Consequently, we get a new hole with the smallest vertex x, and a new promising triple x,t,c with the sum $\sigma(x)+\sigma(c)$ larger than Σ occurs. Therefore, t and b cannot be adjacent and, by Corollary 2.4, $ta\notin E$ as well. Now, to avoid a net $N(a,x,c;b,t,f)$, vertices t and

f have to be adjacent, and by Corollary 2.4, $th \in E$ too. Applying property (P1) to $t < c < h$, $th \in E$, $ch \notin E$ we get a vertex $s > h$ which is adjacent to c and not to t. From $s > h > f > c = ln(a)$ and the choice of the vertex $f = c_b$ we infer $sb \in E$ and $sa \notin E$. Since s is adjacent to b, c and not to a, a contradiction with Corollary 2.4 arises.

Hence, a is the largest vertex of S_c. Assume now that $b \neq u := sn(a)$ and that a, u, c is not a promising triple. Then necessarily, $ub \in E$ (otherwise a claw $K(a; u, b, c)$ arises) and, by Corollary 2.4, $uf \notin E$. Furthermore, the vertex u_a ($u_a > c$, $uu_a \in E$, $au_a \notin E$) is adjacent to c. By Corollary 2.4, u_a cannot be adjacent to b. From the maximality of Σ, u, b, u_a is not a promising triple. Hence, the vertex b_u ($b_u > u_a$, $bb_u \in E$, $ub_u \notin E$) is adjacent to u_a. Since $b_u > u_a > c$ and $c = ln(a)$, vertices b_u and a are not adjacent. Therefore, $b_u c \notin E$ by Corollary 2.4, and we have constructed a claw $K(u_a; u, c, b_u)$.

Thus, a, u, c ($b = u$ is possible) must be a promising triple, and since $c = ln(a)$, a is the largest vertex in S_c and $u = sn(a)$, it is an extreme promising triple. □

Theorem 3.4. *Let G be a CN-free graph of diameter greater than 3. There is a $O(m+n)$ time algorithm that decides whether G has a hole and in the affirmative case finds one.*

Proof. First we find a LexBFS-ordering σ of G. This can be done in linear time. Then, using this ordering σ, in $O(\sum_{v \in V} deg(v)) = O(m+n)$ time we find the vertex $ln(v)$ for every $v \in V$ and create the set F. Now, for every vertex $c \in F$ we do the following. Find in $deg(c) + \sum_{\{v:\ c = ln(v)\}} deg(v)$ time the largest vertex a such that $c = ln(a)$ and there exists a vertex $t \neq c$ with $t > a$, $ta \in E$, $tc \notin E$. Then find the vertex $b = sn(a)$ in time $deg(c) + deg(a)$. Finally, find in $deg(a)$ time the vertex b_a and check in time $deg(c)$ whether b_a and c are adjacent.

After this, if, for every c, b_a was adjacent to c, then, by Lemma 3.3, G does not have any promising triple and, by Lemma 3.2, G does not contain any hole. Otherwise, if, for some $c \in F$, we get $b_a c \notin E$, then a, b, c is a promising triple and we can construct a hole of G in linear time (see Lemma 3.1). Evidently, the total time we spent is $O(m+n)$, since for each vertex $v \in V$ the vertex $ln(v)$ is unique. □

3.2. Finding a dominating pair

Lemma 3.5. *Let G be a CN-free graph which does not contain holes H_k ($k \geqslant 6$) and graphs H_5^+, S_3^- as induced subgraphs. Let also σ be a LexBFS-ordering of G and let a be the vertex that has number 1 in σ. Then, for every $v, u \in V$, $dist(v, u) \leqslant ecc(a)$ holds.*

Proof. Assume by way of contradiction that $dist(v, u) > ecc(a)$, and let P be a shortest path connecting v and u such that the sum of numbers of vertices of P in σ is maximum.

First we will prove that $a \in N[P]$. Suppose $a \notin N[P]$ and consider shortest paths P_v and P_u connecting vertex a with v and u, respectively. Since $dist(a, v) \leqslant ecc(a) < dist(v, u)$ and $dist(a, u) \leqslant ecc(a) < dist(v, u)$, by Lemma 2.5, $P \cap P_v = \{v\}$, $P \cap P_u = \{u\}$.

Moreover, if there is a chord between P and P_v, then it is unique and both its endvertices are adjacent to v. Similarly, both endvertices of the chord (if it exists) between P and P_u are adjacent to u. Lemma 2.5 can also be applied to paths P_v and P_u. Assume, without loss of generality, that $dist(v,a) \geqslant dist(u,a)$. If $u \in \mathrm{N}[P_v]$, then u is adjacent to a vertex of P_v distinct from a and, hence, we will have $dist(v,u) \leqslant dist(v,a)$, that is impossible. Thus, $u \notin \mathrm{N}[P_v]$ and, by Lemma 2.5, $P_u \cap P_v = \{a\}$ and both endvertices of the chord (if it exists) between P_v and P_u are adjacent to a. Consequently, we have an induced subgraph of G similar to that shown in Fig. 3. Again, as in the proof of Lemma 2.6, we will get an induced subgraph of G isomorphic to a hole H_k ($k \geqslant 6$), or to H_5^+, or to S_3^- (note that S_3 cannot arise since $dist(v,u) > ecc(a) \geqslant dist(v,a) \geqslant 2$, i.e. the length of P is at least 3).

Hence, a is in $\mathrm{N}[P]$. We prove that a cannot belong to any shortest path connecting v and u. If a lies on a shortest path P' between v and u, then for neighbors $b,c \in P'$ of a, we have $a < b < c$ (we may assume that $b < c$ holds). Since G does not contain holes H_k ($k \geqslant 6$), by Lemma 3.1, the vertex b_a ($b_a > c$, $bb_a \in E$, $ab_a \notin E$) either is adjacent to c or together with the vertex c_b and vertices a,b,c forms a hole H_5. From $dist(v,u) > ecc(a) \geqslant 2$, we deduce that vertex b or vertex c is distinct from v,u. Let, without loss of generality, $b \neq v,u$. Then, to avoid a claw, b_a must be adjacent to the neighbor of b on P' distinct from a. From distance requirements we conclude that vertices b_a and c were not adjacent and that c_b is not adjacent to the neighbor of b on P' distinct from a. Hence, we obtain an induced subgraph of G isomorphic to H_5^+. Therefore, a does not belong to any shortest path connecting v and u.

Now, $a \notin P$ but is adjacent to a vertex of P. If $au \in E$ then from $dist(v,a) \leqslant ecc(a) < dist(v,u)$ we would have that a belongs to a shortest path between v and u. So, we conclude that a is adjacent neither to u nor to v. Then, since G is claw-free, a is adjacent to exactly two adjacent vertices of P, say b and c. Without loss of generality, assume $b < c$. Denote by d and f the neighbors on P of b and c, respectively. We have $ad, af \notin E$ and distinguish two cases: $d > b$, $d < b$.

If $d > b$, then by Lemma 3.1 applied to $b < \{d,c\}, dc \notin E$, either there is a vertex $t > \{d,c\}$ adjacent to d,c and not to b, or there exist two vertices t and s which together with b,c,d form a hole $H_5 = (b,c,s,t,d,b)$. In the second case, vertex s must be adjacent to f, since otherwise a claw $K(c;b,s,f)$ arises. But now we have an induced H_5^+. In the first case we get a new shortest path $P' = P \setminus \{b\} \cup \{t\}$ between v and u with larger sum.

Let now $d < b$. Applying (P1) to $a < d < c$, $ac \in E$, $dc \notin E$, we will get a new vertex $t > c$ adjacent to d and not to a. We choose vertex t with largest number in σ. Note that, since $b < t$, vertices c and t cannot be adjacent.

If $bt \notin E$, then Lemma 3.1 applied to $d < b < t$ gives a vertex $x > t$ adjacent to b,t and not to d, or two vertices x and y which form together with d,b,t a hole $H_5 = (d,b,y,x,t,d)$. In the last case, if $xc \notin E$, then we get either a claw $K(b;d,y,c)$ or an induced H_5^+, depending on whether vertices y and c are adjacent. On the other hand, if $xc \in E$, then, to avoid a claw $K(c;b,x,f)$, vertex x has to be adjacent to f, and, a new shortest path $P' = P \setminus \{b,c\} \cup \{t,x\}$ with larger sum occurs. In the first case, i.e. when x is adjacent to b and t, to avoid a claw $K(b;d,x,a)$, x must be adjacent to a. Hence, we can apply (P1) to $a < d < x$, $xa \in E$, $xd \notin E$ and get a vertex $s > x$

adjacent to d and not to a. Since $s > x > t$, we arrive to a contradiction with the choice of t.

Thus, vertex b has to be adjacent to t. By Lemma 3.1 applied to $b < c < t$, $ct \notin E$, we will find either a vertex $x > t$ adjacent to t, c and not to b, or two vertices x and y ($t < \{x, y\}$) which form together with b, c, t a hole $H_5 = (b, c, x, y, t, b)$. Note that $xf \in E$, since otherwise we get a claw $K(c; b, f, x)$. Now, in the first case we get a new shortest path $P' = P \setminus \{b, c\} \cup \{t, x\}$ between v and u with larger sum. In the second case we obtain an induced H_5^+ (from $b < c < \{x, y\}$ and the choice of P, vertex d is adjacent neither to x nor to y). □

Corollary 3.6. *Let G be a CN-free graph which does not contain holes H_k ($k \geqslant 6$) and graphs H_5^+, S_3^- as induced subgraphs. Let also σ be a LexBFS-ordering of G and let a be the vertex that has number 1 in σ. Then, $diam(G) = ecc(a)$, and, therefore, both the diameter and a diametral pair of vertices of such a graph can be computed in linear time.*

Proof. If x, y is a diametral pair of vertices of G, then by Lemma 3.5, $diam(G) = dist(x, y) \leqslant ecc(a) \leqslant diam(G)$, i.e $ecc(a) = diam(G)$. Since a LexBFS-ordering of G and the eccentricity of a vertex can be found in linear time, we are done with the diameter. To find a diametral pair of vertices, it is enough to find by BFS a vertex b of G at maximum distance from a. Then a, b is a diametral pair. □

Theorem 3.7. *Let G be a CN-free graph of diameter greater than 3 that does not contain holes H_k ($k \geqslant 6$) as induced subgraphs. Then, a dominating pair of G exists and can be found in linear time.*

Proof. It follows from Corollaries 2.3, 2.7 and 3.6. □

Lemma 3.8. *Let G be a CN-free graph of diameter not greater than 3. There is a linear time algorithm that finds either a dominating pair or an induced doubly dominating cycle of G.*

Proof. First we find a pair of mutually furthest vertices of G. Since $diam(G) \leqslant 3$ this can be done in linear time as follows. For an arbitrary vertex v of G, (using BFS) find a vertex x which is at maximum distance from v, i.e. $dist(v, x) = ecc(v)$. Then, find a vertex y such that $dist(x, y) = ecc(x)$. If $dist(x, y) = ecc(y)$, then x, y are mutually furthest vertices of G. Else, $ecc(y) > ecc(x) \geqslant 1$ must hold. Hence, we continue by finding a new vertex z with $dist(y, z) = ecc(y)$. If $dist(z, y) = ecc(z)$, then z, y are mutually furthest vertices. Otherwise, $ecc(z) > ecc(y) \geqslant 2$, i.e. $ecc(z) = 3$, and the vertex z and a vertex u with $dist(z, u) = ecc(z)$ form a pair of mutually furthest vertices.

Let now a, b be a pair of mutually furthest vertices of G. If $dist(a, b) \leqslant 2$, then either $\mathrm{N}[\{a, b\}] = V$ or there exists a vertex s such that $dist(a, s) = dist(b, s) = 2$. In the first case, a, b form a dominating pair. In the second case, $dist(a, b) = 2$ holds and we get an induced doubly dominating cycle as follows. Let $x \in \mathrm{N}[a] \cap \mathrm{N}[s]$, $y \in \mathrm{N}[b] \cap \mathrm{N}[s]$ and

$v \in N[a] \cap N[b]$. Since G is claw-free none of the edges xb, sv, ay is possible. Hence, vertices a, x, s, y, b, v induce in G either a hole (H_6 or H_5) or a graph isomorphic to S_3 or S_3^-, depending on the existence of the chords xy, xv, yv. By Lemma 2.1, each hole of G doubly dominates G. Furthermore, as we have shown in the proof of Theorem 2.8, either we can construct a new hole (H_6 or H_5) or the induced 4-cycle of S_3^- as well as the inner triangle of S_3 doubly dominate G.

Hence, assume that $dist(a,b) = 3$, and let $C := V \setminus (N[a] \cup N[b])$. First we note that, if (a, v, u, b) is a shortest path between a and b, and c is a vertex of C, then c is adjacent either to both v and u or to none of them (if c is adjacent to only one of v, u, then we will have a claw).

We claim that a, b is a dominating pair if and only if $C \subseteq N[v]$ holds for every vertex v which belongs to a shortest path between a and b, i.e. for every $v \in I(a,b) \setminus \{a,b\}$, where $I(a,b) := \{z \in V\colon dist(a,z) + dist(z,b) = dist(a,b)\}$. Indeed, if a, b is a dominating pair, then for each inner vertex v of a shortest path connecting a and b, $C \subseteq N[v]$ must hold. Suppose now, that $C \subseteq N[v]$ holds for every $v \in I(a,b) \setminus \{a,b\}$, but there exists an induced path P between a and b which does not dominate G. Then necessarily, the length of P is at least 4 and, therefore, we can find a vertex x on P such that $xa, xb \notin E$. This vertex x as well as any vertex $s \notin N[P]$ belongs to the set C. Hence, both s and x are adjacent to a vertex $v \in I(a,b) \cap N(a)$, creating a claw $K(v; a, x, s)$ in G.

Thus, to get a dominating pair or an induced doubly dominating cycle of G, we can proceed as follows. For a given pair of mutually furthest vertices a, b with $dist(a,b) = 3$, first we compute the sets C and $I(a,b)$. This can be done in linear time. Then we check whether $C \subseteq N[v]$ holds for every $v \in I(a,b) \setminus \{a,b\}$. If this holds for every v then, as we have shown, a, b form a dominating pair of G. If this does not hold for some $v \in I(a,b) \setminus \{a,b\}$, then we have a vertex $s \in C$ such that $vs \notin E$. Now, we find a shortest path $P = (a, v, u, b)$, connecting a with b and containing v, and shortest paths $P_a = (a, a', s', s)$ and $P_b = (b, b', s'', s)$ connecting s with a and b, respectively (note that, $a' = s'$ or $b' = s''$ is possible, if the length of P_a or P_b is 2). As usual (see, for example, the proof of Lemma 2.6) we have that $P \cap P_a = \{a\}$, $P \cap P_b = \{b\}$, $P_b \cap P_a = \{s\}$ and only chords $s's'', a'v, b'u$ are possible in the cycle $(a, a', s', s, s'', b', b, u, v, a)$. Therefore, we again have either a hole H_k ($5 \leqslant k \leqslant 9$) or an induced subgraph of G isomorphic to S_3^-, for which we know that its 4-cycle doubly dominates G. □

Theorem 3.9. *There is a* $O(m+n)$ *time algorithm that, for a given CN-free graph* G, *finds either a dominating pair or an induced doubly dominating cycle.*

Proof. First we compute a LexBFS-ordering σ of G. Let u be the vertex that has number 1 in σ. Then, using σ, we look for an extreme promising triple in G. If there is one, then we extend it to a hole H_k ($k \geqslant 5$). By Lemma 2.1, H_k doubly dominates G. If G does not have extreme promising triples, then, by Lemma 3.3, it does not have any promising triples. Hence, by Lemma 3.2, G has no holes or $diam(G) \leqslant 3$. Now we compute the eccentricity of the vertex u. If $ecc(u) \geqslant 4$ then $diam(G) \geqslant 4$ and we can apply Theorem 3.7. Note that, in this case, G has no holes and, therefore, u together with a vertex v, such that $dist(v,u) = ecc(u) = diam(G)$, form a dominating pair. If $ecc(u) \leqslant 3$ then, by Corollary 3.6, the diameter of G cannot be greater than 3. So, we can apply Lemma 3.8 to get a dominating pair or an induced doubly dominating cycle of G with $diam(G) \leqslant 3$. □

4. Concluding remarks

We proved that every (claw, net)-free graph contains an induced doubly dominating cycle or a dominating pair, and that there is a linear time algorithm which, for a given (claw, net)-free graph, finds either a dominating pair or an induced doubly dominating cycle.

Below we show that three classical optimization problems, namely, the *domination problem*, the *independent domination problem* and the *independent set problem*, also can be solved efficiently on (claw, net)-free graphs. The (independent) domination problem asks for an (independent) dominating set of G with the minimum cardinality, while the independent set problem asks for an independent set of G with the maximum cardinality. To solve these problems we use the structural properties of (claw, net)-free graphs, presented in Section 2, and a few known algorithmic results from [22]. In [22], Hempel and Kratsch gave linear time algorithms for all three problems on the class of claw-free AT-free graphs. Note that both the domination problem and the independent domination problem are NP-hard in claw-free graphs. This is due to the fact that they are NP-hard even in line graphs since the edge domination and edge independent domination problems are NP-hard [31]. In contrast, the independent set problem is polynomial time solvable on claw-free graphs [26].

Lemma 4.1. *Let $P=(x_1,\ldots,x_k)$ be an induced path of a CN-free graph G and v be a vertex of G such that $dist(v,P)=2$. Then any neighbor y of v with $dist(y,P)=1$ is adjacent to x_1 or to x_k.*

Proof. Assume that $dist(y,\{x_1,x_k\}) \geqslant 2$, and let x_i be the vertex of P with minimum index i ($i \in \{2,\ldots,k-1\}$) which is adjacent to y. To avoid a claw $K(x_i;y,x_{i-1},x_{i+1})$ we must have $yx_{i+1} \in E$. But then $i+1<k$ and we get either a net $N(y,x_i,x_{i+1};v,x_{i-1},x_{i+2})$ or a claw $K(y;v,x_i,x_{i+2})$ depending on the adjacency of y and x_{i+2}. □

The following important lemma shows that deleting the closed neighborhood of any vertex from a CN-free graph results in an induced subgraph which is AT-free.

Lemma 4.2. *For every vertex v of a CN-free graph G, the graph $G(V \setminus \mathrm{N}[v])$ is a claw-free AT-free graph.*

Proof. Let $G_v := G(V \setminus \mathrm{N}[v])$ be an induced subgraph of G obtained from G by deleting the closed neighborhood of v. Since every hole and every induced subgraph of G which is isomorphic to S_3 or S_3^- dominates G, the graph G_v contains neither holes nor S_3, S_3^- as induced subgraphs.

Assume that G_v contains an asteroidal triple x,y,z. Denote by P_{xy} an induced path connecting vertices x,y and avoiding the (closed) neighborhood of z. Similarly, we can define induced paths P_{xz} and P_{yz}.

Let now t be the closest to z vertex of P_{xz} which has a neighbor on P_{xy}. Then, the neighbor s of t on the subpath of P_{xz} between t and z has no neighbors on P_{xy}. Therefore, we can apply Lemma 4.1 and infer $tx \in E$ (note that $ty \notin E$ since P_{xz} avoids the neighborhood of y).

Thus, among the vertices of $P_{xz} \setminus \{x\}$ only the neighbor of x can be adjacent to a vertex of P_{xy} and, by symmetry, among the vertices of $P_{xy} \setminus \{x\}$ only the neighbor of x can be adjacent to a vertex of P_{xz}. That is, $P_{xy} \cap P_{xz} = \{x\}$ and between these paths only a chord with both endvertices adjacent to x is possible. Since similar facts hold for other path combinations (for P_{xz} with P_{yz} and for P_{xy} with P_{yz}), a subgraph of G_v formed by vertices of $P_{xy} \cup P_{xz} \cup P_{yz}$ will contain either a hole, or S_3, or S_3^- as an induced subgraph (see the proof of Lemma 2.6). But, as we have mentioned above, these induced subgraphs are forbidden for the graph G_v. The contradiction obtained shows that G_v cannot contain asteroidal triples, i.e., it is an AT-free graph. □

Theorem 4.3. *Let G be a CN-free graph with the minimum vertex-degree δ. There are* $O(\delta m)$ *time algorithms for computing a minimum* (*independent*) *dominating set and a maximum independent set of G.*

Proof. We will give a method how to find a minimum independent dominating set D of G. Since every claw-free graph has a minimum dominating set that is independent (see [1]), the set D will be a minimum dominating set of G as well. Moreover, it will be rather clear from our method how one can apply it in order to solve the maximum independent set problem on G.

The method is based on three observations.

(1) Evidently, $N[v] \cap D \neq \emptyset$ for every vertex $v \in V$ and every dominating set D of G.
(2) Assume that x belongs to a minimum independent dominating set of G and D' is a minimum independent dominating set of the graph $G_x = G(V \setminus N[x])$. Then, $D := D' \cup \{x\}$ is a minimum independent dominating set of G.

Proof. Evidently, D is an independent dominating set of G. Assume that D is not a minimum one and let D^* be a minimum independent dominating set of G containing the vertex x. We have $|D^*| < |D|$. Consider the set $D'' := D^* \setminus \{x\}$. This set is independent and dominates $G(V \setminus N[x])$. Hence, D'' is an independent dominating set of G_x but, due to $|D''| = |D^*| - 1 < |D| - 1 = |D'|$, its cardinality is smaller than the cardinality of a minimum independent dominating set D' of G_x. □

(3) Since $G_x = G(V \setminus N[x])$ is a claw-free AT-free graph, we can apply a linear time algorithm from [22] to get a minimum independent dominating set D' of G_x.

So, to compute a minimum independent dominating set of a CN-free graph G in $O(\delta m)$ time, we can proceed as follows. Take a vertex v of G with minimum degree δ. Compute for each vertex $x \in N[v]$ a minimum independent dominating set D_x of the graph $G_x := G(V \setminus N[x])$. Choose a smallest set D_{x^*} from $\{D_x\colon x \in N[v]\}$. Output the set $D_{x^*} \cup \{x^*\}$, it is a minimum independent dominating set of G. □

To our knowledge, this is the first polynomial time algorithm for the (independent) domination problem on CN-free graphs. For the independent set problem, the most efficient previously known algorithm had the complexity $O(n^3)$ [21] and used struction—a completely different method.

Many interesting problems remain still open for CN-free graphs. Among them are the efficient recognition, the connected and total domination problems, the Steiner tree problem, and many others. A natural question to ask is "Can one use the fact, that for each vertex x of a CN-free graph G the graph $G_x = G(V \setminus N[x])$ is claw-free AT-free, to solve on G efficiently further problems which are easy to solve on claw-free AT-free graphs?". More generally, one can define the class of "circular AT-free graphs". A graph G is said to be a circular AT-free graph iff deleting the closed neighborhood of any vertex from G results in an induced subgraph which is AT-free. Can one use the approach described in the proof of Theorem 4.3 to provide, on circular AT-free graphs, efficient solutions for problems solvable on AT-free graphs? At least for the independent domination and independent set problems this works: both these problems are polynomial time solvable on AT-free graphs [8] and, hence, one can use the arguments explained above.

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