DISCRETE APPLIED MATHEMATICS

Discrete Applied Mathematics 105 (2000) 39–50

# On stable cutsets in graphs

Andreas Brandstädt[a,*], Feodor F. Dragan[b,1], Van Bang Le[a], Thomas Szymczak[a]

[a] *Universität Rostock, FB Informatik, Albert-Einstein-Str. 21, D-18051 Rostock, Germany*
[b] *UCLA Computer Science Department, 3514 Boelter Hall, Los Angeles, CA 90095-1596, USA*

Received 18 November 1998; revised 29 January 2000; accepted 21 February 2000

**Abstract**

We answer a question of Corneil and Fonlupt by showing that deciding whether a graph has a stable cutset is $\mathbb{NP}$-complete even for restricted graph classes. Some efficiently solvable cases will be discussed, too. 

*Keywords*: Stable cutsets; $\mathbb{NP}$-completeness; 1-In-3 3Sat; Brittle graphs; Hole-free graphs; Efficient algorithms

## 1. Introduction

In a graph, a *stable set* (a *clique*) is a set of pairwise non-adjacent (adjacent) vertices. A *cutset* is a set of vertices whose deletion results in a disconnected graph. A *stable cutset* is a cutset which is also a stable set. Stable cutsets in graphs have been discussed by Tucker [18], Corneil and Fonlupt [5] in connection with perfect graphs. In [5], Corneil and Fonlupt proposed the following problem.

Stable Cutset: *Given graph G. Does G have a stable cutset*?

This problem is also mentioned by Chvátal et al. in [4]. In this note we prove that Stable Cutset is $\mathbb{NP}$-complete even for $K_4$-free graphs and for graphs with connectivity number 2. Our results are best possible in the sense that Stable Cutset trivially can be solved in linear time for $K_3$-free graphs and for graphs with connectivity number at most 1. After writing a first version of this note, Cunningham [8] informed us that $\mathbb{NP}$-completeness of the Stable Cutset problem (for line graphs) might be derived also from the result of Chvátal [3] on decomposable graphs. We present this in Section 2. In Sections 4 and 5, we shall discuss Stable Cutset in graphs without long

* Corresponding author.
*E-mail addresses:* ab@informatik.uni-rostock.de (A. Brandstädt), dragan@CS.UCLA.EDU (F.F. Dragan), le@informatik.uni-rostock.de (V.B. Le), szymczak@informatik.uni-rostock.de (T. Szymczak)
[1] Author supported by the German Research Community DFG.

0166-218X/00/$ - see front matter 
PII: S0166-218X(00)00197-9

induced cycles. Among them are HHD-free graphs, brittle graphs, hole-free graphs and AT-free graphs. It turns out that for all these graph classes, Stable Cutset can be solved in polynomial time. (For more information on graph classes considered here, see the survey [2].)

Notice that the related Clique Cutset problem can be solved in polynomial time; see for instance [19].

## 2. Stable cutsets in line graphs

In order to make the paper self-contained, we are going to describe a consequence of Chvátal's results on decomposable graphs to the Stable Cutset problem. A graph is called *decomposable* if its vertices can be colored with two colors in such a way that each color appears on at least one vertex and each vertex $v$ has at most one neighbor having a different color from $v$. In other words, a graph is decomposable if its vertices can be partitioned into two nonempty parts such that the edges connecting vertices from different parts form an induced matching.

**Theorem 1** (Chvátal [3]). *Recognizing decomposable graphs is $\mathbb{NP}$-complete, even if the input is restricted to graphs with maximum degree* 4.

Note that this result is best possible in the sense that decomposable graphs with maximum degree at most 3 can be recognized in polynomial time [3]. We will need the following $\mathbb{NP}$-completeness result.

**Corollary 2.** *Recognizing decomposable graphs is $\mathbb{NP}$-complete, even if the input is restricted to graphs with maximum degree* 4 *and minimum degree at least* 2.

**Proof.** If a graph $G$ with maximum degree 4 has a vertex $v$ of degree 1, then we add to $G$ two new vertices $x_v, y_v$ and make vertices $v, x_v$ and $y_v$ pairwise adjacent. It is easy to see, that the new graph is decomposable if and only if $G$ is decomposable. So, the result follows from Theorem 1. □

Recall that the *line graph* $L(G)$ of a graph $G$ has the edges of $G$ as its vertices, and two distinct edges of $G$ are adjacent in $L(G)$ if they are incident in $G$. The relationship between decomposability and having a stable cutset is

**Proposition 3.** *If $L(G)$ has a stable cutset, then $G$ is decomposable. If $G$ is decomposable and has minimum degree at least* 2, *then $L(G)$ has a stable cutset.*

**Proof.** First, let $S$ be a stable cutset of $L(G)$ and let $A$ be a component of $L(G) - S$. Color the vertices of $G$ which are endvertices of an edge in $A \subset E(G)$ with color red, and color the remaining vertices with color blue. Then $G$ is decomposable by this coloring: If the red vertex $x$ has two blue neighbors $y \neq z$, then at least one of the edges $xy, xz$ is not in $S \subset E(G)$ because $S$ is a stable set in $L(G)$. But then $xy$ or $xz$ would belong to $A$. This is impossible because both $y$ and $z$ are blue vertices. Similarly, no blue vertex can have two red neighbors.

Second, let $G$ have minimum degree $\geqslant 2$, and assume that $G$ is decomposable with a suitable coloring in red and blue vertices. Let $R$ and $B$ be the subgraphs of $G$ induced by the red vertices (resp., the blue vertices). Since $G$ is decomposable and has minimum degree $\geqslant 2$, each of $R$ and $B$ contains at least one edge. Now, the set of all edges in $G$ between $R$ and $B$ form a stable cutset in $L(G)$ separating $E(R)$ and $E(B)$. (Note that the condition on minimum degree is necessary: The star $K_{1,n}$ is decomposable, while its line graph $L(K_{1,n}) = K_n$ does not have a stable cutset.) □

From Corollary 2 and Proposition 3 we conclude:

**Theorem 4.** Stable Cutset *is* $\mathbb{NP}$*-complete, even if the input is restricted to line graphs with maximum degree at most* 6.

It is an open question whether the restriction on maximum degree in Theorem 4 is best possible. However, we remark that Stable Cutset is polynomial if the input is restricted to line graphs with maximum degree at most 3.

In [5], Corneil and Fonlupt also asked for the complexity of Stable Cutset in perfect graphs: Given a perfect graph $G$, does $G$ have a stable cutset? The answer follows from the result of Moshi in [15].

**Theorem 5** (Moshi [15]). *Recognizing decomposable graphs is* $\mathbb{NP}$*-complete, even if the input is restricted to bipartite graphs of minimum degree* 2.

From this theorem and Proposition 3 and the well-known fact that line graphs of bipartite graphs are perfect [2], we can conclude:

**Theorem 6.** Stable Cutset *is* $\mathbb{NP}$*-complete, even if the input is restricted to line graphs of bipartite graphs, and thus to perfect graphs.*

## 3. Stable cutsets in $K_4$-free graphs

Let $K_n$ denote a complete graph with $n$ vertices. In this section we show that Stable Cutset is $\mathbb{NP}$-complete for $K_4$-free graphs. This result is best possible in the sense that for $K_3$-free graphs, Stable Cutset can be easily solved in linear time: If $G$ is $K_3$-free and has at least three vertices, then for every vertex $v$ of $G$, $\{v\}$ or the neighborhood of $v$ is a stable cutset of $G$.

**Theorem 7.** *It is* $\mathbb{NP}$*-complete to decide whether a given* $K_4$*-free graph has a stable cutset.*

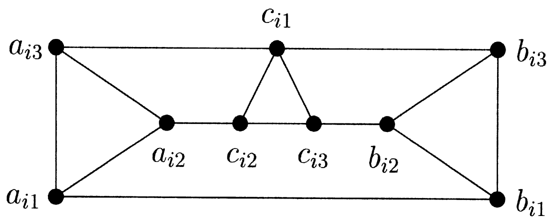


Fig. 1. The graph $G(C_i)$.

**Proof.** We shall reduce the following problem, which was proved to be $\mathbb{NP}$-complete by Schaefer [16], to Stable Cutset.

1-In-3 3Sat (without negative literals). *Let $\mathscr{C}$ be a collection of $m$ clauses over the set $V$ of $n$ Boolean variables such that every clause has exactly three variables. Is there a truth assignment satisfying $\mathscr{C}$ such that each clause in $\mathscr{C}$ has exactly one true variable*?

For each variable $v \in V$ we take a labelled vertex $v$. For each clause $C_i = c_{i1} \vee c_{i2} \vee c_{i3}$, where $c_{ij}$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant 3)$ are variables taken from $V$, define the labelled graph $G(C_i)$ as shown in Fig. 1.

Moreover, we consider a $K_3$ $R = r_1 r_2 r_3$ and a $K_2$ $T = t_1 t_2$. We now create the graph $G = G(\mathscr{C})$ from the labelled vertices $v$ $(v \in V)$, the graphs $G(C_i)$ $(1 \leqslant i \leqslant m)$, the graphs $R, T$, and edges

$$vc_{ij} \text{ if and only if } c_{ij} \text{ is the variable } v \ (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant 3),$$
$$vr_1, vr_2 (v \in V),$$
$$r_1 a_{i1}, r_2 b_{i1}, r_3 a_{i1}, r_3 b_{i1} \ (1 \leqslant i \leqslant m),$$
$$t_1 c_{i1}, t_1 c_{i2}, t_2 c_{i1}, t_2 c_{i3} \ (1 \leqslant i \leqslant m).$$

Clearly, $G$ has no $K_4$. We now are going to show that 1-In-3 3Sat is satisfied if and only if $G$ has a stable cutset.

*Suppose that there is a truth assignment satisfying* 1-In-3 3Sat.

Then a stable cutset $S$ of $G$ can be constructed as follows:

(S1) $S := \{v : v \text{ false}\} \cup \{c_{ij} : c_{ij} \text{ true}\}$,

(S2) For $1 \leqslant i \leqslant m$:

If $c_{i1} \in S$, put $a_{i1}, b_{i2}$ into $S$,
If $c_{i2} \in S$, put $a_{i3}, b_{i1}$ into $S$,
If $c_{i3} \in S$, put $a_{i1}, b_{i3}$ into $S$.

Since exactly one of $c_{i1}, c_{i2}, c_{i3}$ is true, $S$ is a stable set after step (S1). By definition of the $G(C_i)$s, $S$ remains a stable set after (S2). From (S1) and (S2) it is easy to see that $G - S$ splits into exactly two connected components; one contains $R$ and the other contains $T$.

*Suppose that $G$ has a stable cutset $S$.*

Then a truth assignment for 1-In-3 3Sat can be defined as follows:

$v$ is true if $v \notin S$ and false, otherwise.

We now are going to show that every clause has exactly one true literal by this assignment. First, as $S$ is a stable set, there are at least two vertices $r, r'$ in $R - S$. Also, there is a vertex $t$ in $T - S$. By construction of $G$,

$$\text{every vertex in } V \text{ is adjacent to } r \text{ or } r'. \tag{1}$$

Next, it is easy to see that

$$\text{for each vertex } w \in G(C_i) - S, \text{ there is a path in } (G(C_i) - S) \cup \{r, r', t\} \text{ connecting } w \text{ and } \{r, r'\}, \text{ or } w \text{ and } t. \tag{2}$$

Hence, we can conclude that

$$\text{there is no path in } G - S \text{ connecting } t \text{ and } \{r, r'\}, \tag{3}$$

otherwise by (1) and (2), $G - S$ would be connected. (3) implies that

$$\text{for each } i, \text{ at least one, hence exactly one of } c_{i1},\ c_{i2},\ c_{i3} \text{ belongs to } S. \tag{4}$$

Moreover, for each $v \in V$ with $v = c_{ij}$, we have in $G$:

$$v \in S \text{ if and only if } c_{ij} \notin S. \tag{5}$$

The only if-part of (5) is clear because $S$ is a stable set. To see the if-part, assume that $c_{ij} \notin S$. If $v \notin S$, then there is a path in $G - S$ connecting $t$ and $\{r, r'\}$ with $c_{ij}$ and $v$ as its inner vertices. This contradicts (3).

Now, by (4), every clause has exactly one literal in $S$. By (5), this literal is the only true variable of that clause by our assignment. The proof of Theorem 7 is complete. □

**Remark.** By definition, a graph is *k-connected* if it has no cutset of less than $k$ vertices. Our graph $G$ in the proof of Theorem 7 is 3-connected, showing that Stable Cutset is $\mathbb{NP}$-complete for 3-connected graphs. We are going to describe a stronger fact. The *connectivity number* of an incomplete graph is the minimum cardinality of a cutset of that graph; the connectivity number of the complete graph $K_n$ is $n-1$. Clearly, Stable Cutset is easy for graphs with connectivity number at most 1 (separable graphs).

The following simple transformation shows that Stable Cutset is already $\mathbb{NP}$-complete for graphs with connectivity number 2: Consider a 3-connected graph $G$, and let $xy$ be an arbitrary edge of $G$. Let $G'$ be the graph obtained from $G$ by taking a new vertex $v$ and adding exactly two new edges $vx$ and $vy$. Clearly, $G'$ has connectivity number 2 and it is easy to see that $G$ has a stable cutset if and only if $G'$ has a stable cutset.

## 4. Stable cutsets in HHD-free and brittle graphs

*Holes* are chordless cycles of length at least five, a *house* is the complement of a chordless path with five vertices, a *domino* is a bipartite graph consisting of a cycle of length six with exactly one chord. Graphs without holes, house and domino are called *HHD-free* graphs; they are introduced in [11], and generalize the well understood triangulated graphs. Notice that by a result due to Dirac [9], no 2-connected triangulated graph has a stable cutset.

A cutset $S$ of a graph $G$ is *minimal* if no proper subset of $S$ is a cutset of $G$. A set of vertices $H$ of $G$ is called *homogeneous* if $H$ consists of at least two but not all vertices of $G$ and every vertex outside $H$ is adjacent to all vertices or to no vertex in $H$. A homogeneous set $H$ of $G$ is *maximal* if there is no homogeneous set containing $H$ properly.

**Theorem 8.** *Let $G$ be a 2-connected HHD-free graph. Every minimal stable cutset in $G$ is a maximal homogeneous set.*

Since the maximal homogeneous sets of a graph can be found in linear time [7,14], Theorem 8 implies that Stable Cutset is easy for HHD-free graphs. Moreover, Theorem 8 is interesting for HHD-free graphs in its own right.

**Proof of Theorem 8.** Let $S$ be a minimal stable cutset in $G$. It is easy to see that

$$\text{every vertex of } S \text{ has a neighbor in every component of } G - S. \tag{6}$$

(Actually, (6) holds for every minimal cutset $S$ in an arbitrary graph $G$.) We first show that $S$ is homogeneous. Suppose to the contrary that $S$ is not a homogeneous set in $G$. Then there is a component $A$ of $G-S$ and a vertex $a \in A$ such that $a$ is adjacent to a vertex $x \in S$ but not to $y \in S$. Let $a' \in A$ be a neighbor of $y$ (see (6)) such that a path $P$ in $A$ connecting $a$ and $a'$ is of shortest length. Note that $|E(P)| \geqslant 1$. Consider a component $B \neq A$ of $G-S$ and let $b$ and $b'$ be a neighbor of $x$ and, respectively, a neighbor of $y$ in $B$ (see (6)) such that a path $Q$ connecting $b$ and $b'$ in $B$ is of shortest length. Let $a''$ be the first neighbor of $x$ on $P$ (from $a'$ to $a$), and let $P'$ be the subpath of $P$ between $a'$ and $a''$. By these choices, $P', Q$, $x$ and $y$ form a chordless cycle of length $4 + |E(P')| + |E(Q)|$, implying $|E(P')| = |E(Q)| = 0$. Thus $a'' = a'$ and $b = b'$, and therefore $a'xbya'$ is an induced cycle. Now, by considering the path $P$ we get a hole, a domino or a house. This contradiction shows that $S$ must be a homogeneous set. Theorem 8 now follows from the following general observation which is easy to see: If $S$ is a minimal cutset and a homogeneous set as well in an arbitrary graph $G$, then $S$ is a maximal homogeneous set in $G$. □

HHD-free graphs form a particular subclass of the class of brittle graphs introduced by Chvátal and studied by Hoàng and Khouzam [11]. To give the definition of brittle graphs we need some notions.

We write $P_n = v_1 v_2 \dots v_n$ for the chordless path with $n$ vertices $v_1, \dots, v_n$ and $n-1$ edges $v_i v_{i+1}$ $(1 \leqslant i \leqslant n-1)$. The vertices $v_1$, $v_n$ are the *endpoints*, and the inner vertices $v_2, \dots, v_{n-1}$ are the *midpoints* of the path $P_n$.

A vertex $v$ in a graph $G$ is called *simplicial* if the neighborhood $N(v)$ in $G$ induces a clique; $v$ is called *co-simplicial* if it is simplicial in the complement $\bar{G}$ of $G$.

Clearly, a vertex is simplicial if and only if it is not the midpoint of any $P_3$. A vertex $v$ in $G$ is then called *semi-simplicial* if $v$ is not a midpoint of any $P_4$ in $G$. A graph $G$ is called *brittle* if, for each induced subgraph $F$ of $G$, $F$ or $\bar{F}$ has a semi-simplicial vertex. In [11] it is proved that every HHD-free graph is brittle. Moreover,

every brittle graph has a simplicial vertex or a co-simplicial vertex or a homogeneous set. (7)

We now are going to show that Stable Cutset is easy for brittle graphs. In doing this, we first discuss stable cutset in graphs having a simplicial, respectively, a co-simplicial vertex.

**Lemma 9.** *Let $v$ be a simplicial vertex in a graph $G$.*
(i) *If* $\deg(v) = 1$ *then $G$ has a stable cutset if and only if* $|V| \geqslant 3$.
(ii) *If* $\deg(v) \geqslant 2$ *then $G$ has a stable cutset if and only if $G - v$ has a stable cutset.*

**Proof.** (i) is clear because $N(v)$ is a stable cutset if $|V| \geqslant 3$. We now prove (ii). Let $c(G)$ denote the number of connected components of $G$ and set $G' := G - v$.

First, suppose $S$ is a stable cutset in $G$. If $v \notin S$ we get $c(G-S) = c(G'-S)$ since $N(v)$ is a clique and $N(v) \setminus S \neq \emptyset$. Therefore $S$ is a stable cutset in $G'$, too. Now assume that $v \in S$. Then $N(v) \cap S = \emptyset$ and thus $S' := S \setminus \{v\}$ is a stable cutset in $G$ implying that $S'$ is a stable cutset in $G'$.

Now, for the other direction let $S'$ be a stable cutset in $G'$. Since $N(v)$ is a clique and $\deg(v) \geqslant 2$, $N(v) \setminus S \neq \emptyset$ implying that $c(G-S) = c(G'-S)$. □

**Lemma 10.** *Let $v$ be a co-simplicial vertex in a graph $G$. If $G$ has a stable cutset then*
(i) *$G - N(v)$ is a stable cutset, or*
(ii) *$N(w)$ is a stable cutset for some vertex $w$ in $G - N(v) - v$.*

**Proof.** Note that $G - N(v)$ is a stable set because $v$ is co-simplicial. Thus, if $N(v)$ induces a disconnected graph then $G - N(v)$ is a stable cutset, and we get (i). Therefore we may assume that

$N(v)$ induces a connected graph. (8)

Note that for every $w \in G - N(v) - v$,

$N(w) \subseteq N(v)$. (9)

Thus, if $N(w)$ is a stable set for some vertex $w$ in $G - N(v) - v$, then $N(w)$ is clearly a stable cutset and we get (ii). Therefore, we may assume further that

$$\text{for every } w \in G - N(v) - v,\ N(w) \text{ contains an edge (in } N(v)). \tag{10}$$

Now consider a stable cutset $S$ of $G$. (8) and (9) imply that $v \notin S$. By (10), every vertex $w$ outside $S$ is connected to $v$. Thus $G - S$ is connected, a contradiction. We have shown that (i) or (ii) must hold. □

Next, we consider the reduction of homogeneous sets.

**Lemma 11.** *Let $G = (V, E)$ be a connected graph and $H$ a proper homogeneous set in $G$.*

(i) *Let $h \in H$. If $H$ is additionally stable then $G$ has a stable cutset if and only if $G - (H - h)$ has a stable cutset.*
(ii) *If $H$ contains an edge $h_1h_2$ then $G$ has a stable cutset if and only if $G - (H \setminus \{h_1, h_2\})$ has a stable cutset.*

**Proof.** (i) Let $S$ be a stable cutset in $G$. If $H \subseteq S$ then $(S \setminus H) \cup \{h\}$ is a stable cutset in $G' := G - (H - h)$. If $H \nsubseteq S$ then $S \setminus H$ is a stable cutset in $G$, too, yielding that $S \setminus H$ is a stable cutset in $G'$. For the other direction let $S'$ be a stable cutset in $G'$. If $h \notin S'$ then clearly $S'$ is a stable cutset in $G$, too. Otherwise, since $H$ is stable $(S' \setminus \{h\}) \cup H$ is a stable cutset in $G$.

(ii) If $S$ is a stable cutset in $G$ then $S \setminus H$ is a stable cutset in $G' := G - (H \setminus \{h_1, h_2\})$. If $S'$ is a stable cutset in $G'$ then $S' \setminus \{h_1, h_2\}$ is a stable cutset in $G$. □

As a consequence of the Lemmas 9–11 we can now prove

**Theorem 12.** Stable Cutset *can be solved in polynomial time for brittle graphs.*

**Proof.** Let $G$ be a brittle graph. If $G$ contains a simplicial vertex or a homogeneous set which is not an edge we can reduce in polynomial time the problem to a smaller brittle graph (see Lemmas 9 and 11). If $G$ contains a co-simplicial vertex then we are done by Lemma 10. Therefore, we suppose that $G$ contains no simplicial and no co-simplicial vertices and that every homogeneous set induces an edge.

Let $G'$ be the graph obtained from $G$ by contracting every homogeneous set to a representing vertex. Clearly, $G'$ contains no homogeneous set and is an induced subgraph of $G$. In particular, $G'$ is brittle. By (7), $G'$ must contain a simplicial or co-simplicial vertex. Since a simplicial vertex in $G'$ is also simplicial in $G$,

there exists a co-simplicial vertex $v$ in $G'$.

Now, we show that if $G$ has a stable cutset then one of the following sets must be a stable cutset in $G$:

(a) $N(w)$ for some vertex $w \in G - N(v) - v$, or

(b) $M \cup \{v\}$ where $M$ consists of all vertices in $G - N(v) - v$ not contained in a homogeneous set and moreover $M$ contains one representative from every homogeneous set.

The proof is similar to that of Lemma 10. □

## 5. Stable cutsets in hole-free graphs and related classes

In this section we shall show that Stable Cutset is still easy for a larger class than the brittle graphs, namely for the class of hole-free graphs.

Let us call a graph *k-chordal* if it has no chordless cycle of length at least $k$. Thus triangulated graphs are exactly the 4-chordal graphs, and hole-free graphs are exactly the 5-chordal graphs. Note that $k$-chordality can be recognized in polynomial time for every fixed $k$ [17]. By our result, it should be interesting to investigate the complexity of Stable Cutset for $k$-chordal graphs for fixed $k \geqslant 6$. For 6-chordal graphs we get the following result.

For a subset $M \subseteq V$ we denote by $N(M)$ the neighborhood of $M$ in $G$; i.e., all vertices from $V \setminus M$ which have a neighbor in $M$.

**Lemma 13.** *Let $G$ be a connected 6-chordal graph. If $G$ has a stable cutset then it has a stable cutset that is the intersection of the neighborhoods of two cliques.*

**Proof.** Let $S$ be a minimal stable cutset in $G$, and $A$ and $B$ two different connected components of $G - S$. Let $C$ be a clique from $A$ with maximum number of neighbors in $S$. We are going to show that $S \subseteq N(C)$. Assuming the contrary, there exists a vertex $y \in S$ that has no neighbor in $C$. Recall that, by the minimality of the cutset $S$, every vertex from $S$ has a neighbor in $A$ and a neighbor in $B$. Let $a_0 \in A - C$ be a neighbor of $y$ such that a path $P$ in $A$ connecting $a_0$ and $C$ is of shortest length $k \geqslant 1$. Write $P = a_0 a_1 \ldots a_k$ with $a_k \in C$ and $a_i \notin C$ for all $i \neq k$.

**Claim.** $k = 1$.

**Proof of the Claim.** By the choice of $C$, $C$ has a neighbor $x \in S$ such that $x$ is nonadjacent to both $a_0$ and $a_1$ (otherwise, the clique $\{a_0, a_1\}$ from $A$ would have more neighbors in $S$ than $C$). Let $a$ be a vertex from $C$ adjacent to $x$, and let $b$ be a neighbor of $x$ in $B$ and $b'$ be a neighbor of $y$ in $B$ such that a path $P'$ in $B$ connecting $b$ and $b'$ is of shortest length; $b = b'$ is possible. Note, that $x$ cannot be adjacent to any vertex of $P$. Otherwise, let $i$ be minimal such that $xa_i$ is an edge. Then $i \geqslant 2$ and $xa_i \ldots a_0 y P' x$ is an induced cycle of length $i + 4 + |E(P')| \geqslant 6$, a contradiction. Now, let $j$ be minimal such that $a$ is adjacent to $a_j$. By the choice of $P$, $j = k - 1$ or $j = k$, hence the induced

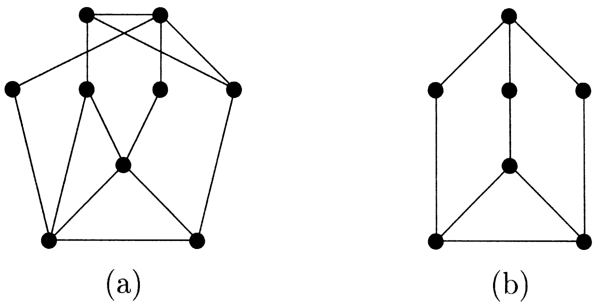


Fig. 2. Two special 6-chordal graphs.

cycle $xaa_j \ldots a_0 y P' x$ is of length $j+5+|E(P')| \geqslant k+4$. Since $G$ is 6-chordal, $k=1$. The claim is proved. □

Thus $P=a_0a_1$ with $a_0 \notin C$ and $a_1 \in C$. But then

$$Q{:=}\{a_0,a_1\} \cup \{a \in C\colon \exists x \in N_S(a) \text{ such that } x \text{ is nonadjacent to } a_0 \text{ and } a_1\}$$

is a clique (otherwise $G$ would have an induced cycle of length $\geqslant 6$). By definition of $Q$, every neighbor of $C$ in $S$ is also a neighbor of $Q$. Since $y \notin N(C)$, $Q$ therefore has more neighbors in $S$ than $C$. This contradiction proves $S \subseteq N(C)$.

By symmetry, a clique $C'$ from $B$ with maximum number of neighbors in $S$ satisfies $S \subseteq N(C')$. Since $S$ is a cutset, and $C$ and $C'$ are in different connected components of $G-S$, $S=N(C) \cap N(C')$. □

From this lemma we immediately conclude

**Theorem 14.** Stable Cutset *can be solved in polynomial time for* 6-*chordal graphs with constant bounded clique size, in particular for* 6-*chordal* $K_4$-*free graphs.*

The graph (a) from Fig. 2 is a 6-chordal graph with a stable cutset of the form $N(K_3) \cap N(K_2)$.

**Theorem 15.** *Let* $G=(V,E)$ *be a connected* 6-*chordal graph which does not contain an induced subgraph isomorphic to the graph* (b) *from Fig* 2. *If G has a stable cutset then it has a stable cutset that is the intersection of the neighborhoods of two elements from* $V \cup E$.

**Proof.** Let $S$ be a minimal stable cutset in $G$, and $A$ and $B$ two different connected components of $G-S$. Since $G$ is a 6-chordal graph there exist two cliques $C \subseteq A$ and $C' \subseteq B$ such that $N(C) \cap N(C')=S$. We may choose $C$ and $C'$ minimal by inclusion. Then every vertex $a$ from $C$ has a neighbor $x_a$ in $S$ (we call it a *personal neighbor*) that is adjacent only to $a$ from $C$. Now let $|C| \geqslant 3$ and $a,b,c$ be three vertices of $C$ with personal neighbors $x_a, x_b, x_c$ in $S$. We have $x_v v \in E$ and $x_v u \notin E$ for $v, u \in \{a, b, c\}$, $u \neq v$. Since $G$ is a 6-chordal graph as before we can show that every two vertices from $\{x_a, x_b, x_c\}$ have a common neighbor in $B$. If we assume that there is no common neighbor of all three vertices in $B$ then we will get an induced cycle of length 6. Hence, there must be a common neighbor $w$ of $x_a, x_b, x_c$ in $B$ and we have constructed an induced subgraph of $G$ isomorphic to the graph (b) from Fig. 2. □

Since the graph (b) in Fig. 2 contains a chordless cycle of length 5 we derive

**Corollary 16.** *Let G be a connected hole-free graph. If G has a stable cutset then it has a stable cutset that is the intersection of the neighborhoods of two vertices. Thus,* Stable Cutset *can be solved in polynomial time for hole-free graphs.*

A graph $G$ is called $(k, l)$-*chordal* (see [1]) if every cycle of length greater than $k-1$ has at least $l$ chords. Since the graph (b) in Fig. 2 contains a cycle of length 6 with exactly one chord we conclude

**Corollary 17.** Stable Cutset *can be solved in polynomial time for* $(6, 2)$-*chordal graphs.*

A set of three vertices of a graph is called an *asteroidal triple* [13] if every two of them can be connected by a path avoiding the closed neighborhood of the third vertex. A graph $G$ is called *AT-free* [6] if it contains no asteroidal triples. Since the graph (b) in Fig. 2 contains an asteroidal triple and AT-free graphs are 6-chordal we have the following

**Corollary 18.** Stable Cutset *can be solved in polynomial time for AT-free graphs.*

We have learnt from [10] that in [12] the $\mathbb{NP}$-completeness of Stable Cutset has been derived from Chvátal's result on decomposable graphs.

## References

[1] G. Ausiello, A. D'Atri, M. Moscarini, Chordality properties on graphs and minimal conceptual connections in semantic data models, J. Comput. System Sci. 33 (1986) 179–202.
[2] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications, Vol. 3, SIAM, Philadelphia, PA, 1999.
[3] V. Chvátal, Recognizing decomposable graphs, J. Graph Theory 8 (1984) 51–53.
[4] V. Chvátal et al., Collection of problems concerning perfect graphs, Dimacs Workshop on Perfect Graphs, Princeton, 1993, `ftp://dimacs.rutgers.edu/pub/perfect/problems.tex`.
[5] D.G. Corneil, J. Fonlupt, Stable set bonding in perfect graphs and parity graphs, J. Combin. Theory Ser. B 59 (1993) 1–14.
[6] D.G. Corneil, S. Olariu, L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997) 399–430.
[7] A. Cournier, M. Habib, A new linear time algorithm for modular decomposition, LIRMM, University Montpellier, 1995.

[8] W.H. Cunningham, personal communication, May 1998.
[9] G. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71–76.
[10] T. Feder, P. Hell, S. Klein, R. Motwani, Complexity of graph partition problems, preprint, 1999.
[11] C.T. Hoàng, N. Khouzam, On brittle graphs, J. Graph Theory 12 (1988) 391–404.
[12] S. Klein, C.M.H. de Figueiredo, The NP-completeness of multi-partite cutset testing, Congr. Numer. 119 (1996) 217–222.
[13] C.G. Lekkerkerker, J.Ch. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. 51 (1962) 45–64.
[14] R.M. McConnell, J. Spinrad, Linear-time modular decomposition and efficient transitive orientation of comparability graphs, Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, 1994, pp. 536–545, Discrete Math. 201 (1999) 189–241.
[15] A.M. Moshi, Matching cutsets in graphs, J. Graph Theory 13 (1989) 527–536.
[16] T.J. Schaefer, The complexity of the satisfability problem, Proceedings of the 10th Annual ACM Symposium on Theory of Computing, 1978, pp. 216–226.
[17] J. Spinrad, Finding large holes, Inform. Process. Lett. 39 (1991) 227–229.
[18] A. Tucker, Coloring graphs with stable cutsets, J. Combin. Theory Ser. B 34 (1983) 258–267.
[19] S.H. Whitesides, An algorithm for finding clique cut-sets, Inform. Process. Lett. 12 (1981) 31–32.