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New Graph Classes of Bounded Clique-Width*

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Abstract. The clique-width of graphs is a major new concept with respect to the efficiency of graph algorithms; it is known that every problem expressible in a certain kind of Monadic Second Order Logic called *LinEMSOL*($\tau_{1,L}$) by Courcelle et al. is linear-time solvable on any graph class with bounded clique-width for which a k-expression for the input graph can be constructed in linear time. The notion of clique-width extends the one of treewidth since bounded treewidth implies bounded clique-width. We give a complete classification of all graph classes defined by forbidden one-vertex extensions of the P_4 (i.e., the path with four vertices a, b, c, d and three edges ab, bc, cd) with respect to bounded clique-width. Our results extend and improve recent structural and complexity results in a systematic way.

1. Introduction

Recently, in connection with graph grammars, Courcelle et al. in [17] introduced the notion of the clique-width of a graph which is closely related to modular decomposition.

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The concept of the clique-width of a graph has attracted much attention due to the fact that, in [18], Courcelle et al. have shown that every graph problem definable in LinEMSOL($\tau_{1,L}$) (a variant of Monadic Second Order Logic) is linear-time solvable on graphs with a bounded clique-width if not only the input graph G but also a k-expression of G is given. (The time complexity may be sublinear with respect to the size of the input graph G if the input is just a k-expression of G). The problems Vertex Cover, Maximum Weight Stable Set (MWS), Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems.

It is known that the class of P_4-free graphs (also called *cographs*) is exactly the class of graphs having clique-width at most 2, and a 2-expression can be found in linear time along the cotree of a cograph. Due to the basic importance of cographs, it is of interest to consider graph classes defined by forbidden one-vertex extensions of P_4—see Figure 2—which are natural generalizations of cographs.

The aim of this paper is to investigate the structure and clique-width of all such graph classes in a systematic way. This leads to a complete classification of these classes with respect to bounded clique-width. Part of the motivation comes from known examples such as the (P_5,co-P_5,bull)-free graphs studied by Fouquet in [22] (see Theorem 12) and the (P_5,co-P_5,chair)-free graphs studied by Fouquet and Giakoumakis in [23] (see Theorem 8) and [24]. Moreover, there are papers such as [33] and [38] dealing with (chair,co-P,gem)-free graphs and [26] dealing with (P_5,P,gem)-free graphs where it is shown that the MWS problem can be solved in polynomial time on these classes. Our results imply bounded clique-width and linear time for any LinEMSOL($\tau_{1,L}$) definable problem including MWS for these classes as well as for many other examples. This also continues research done in [2], [4]–[7] and [9]–[12] and is partially based on some of the results of these papers. The results of this paper were first presented in [3].

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. For a vertex $v \in V$, let $N(v) = \{u \mid uv \in E\}$ denote the *neighborhood* of v in G, and for a subset $U \subseteq V$ and a vertex $v \notin U$, let $N_U(v) = \{u \mid u \in U, uv \in E\}$ denote the *neighborhood* of v with respect to U.

Two disjoint vertex sets X, Y form a *join*, denoted by X ① Y (*co-join*, denoted by X ⓪ Y) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds.

Subsequently, we consider join and co-join also as operations, i.e., the join operation between disjoint vertex sets X, Y adds all edges between them whereas the co-join operation for X and Y is the disjoint union of the subgraphs induced by X and Y (without edges between them).

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. A vertex set $M \subseteq V$ is a *module* if no vertex from $V \backslash M$ distinguishes two vertices from M, i.e., every vertex $v \in V \backslash M$ has either a join or a co-join to M. A module is *trivial* if it is either the empty set, a one-vertex set or the entire vertex set V. Nontrivial modules are called *homogeneous sets*. A graph is *prime* if it contains only trivial modules. The notion of a module plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms—see, e.g., [36] for modular decomposition of discrete structures and its algorithmic use.

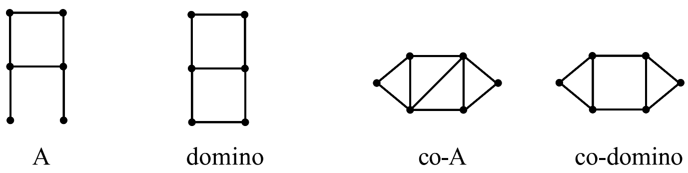

Fig. 1. The A and domino and their complements.

For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.

A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. Let co-$G = \overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k - 1$ edges, and for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* in G is an induced subgraph of G isomorphic to C_k, $k \geq 5$. An *odd hole* is a hole with an odd number of vertices. Note that the P_4 is the smallest nontrivial prime graph with at least three vertices and the complement of a P_4 is a P_4 itself. The *house* is the $\overline{P_5}$.

The *domino* (see Figure 1) is the graph consisting of a P_5 with vertices a, b, c, d, e and edges ab, bc, cd, de plus an additional vertex f being adjacent to a, c and e. The graph A (see Figure 1 as well) is the graph consisting of a C_4 with vertices a, b, c, d and edges ab, bc, cd, da plus two additional vertices e, f such that e is adjacent to c and f is adjacent to d.

For a subgraph H of G, a vertex not in H is a *k-vertex for H* if it has exactly k neighbors in H. We say that H *has no k-vertex* if there is no k-vertex for H. The subgraph H *dominates* the graph G if there is no 0-vertex for H in G.

See Figure 2 for the definition of chair, P, bull, gem and their complements. Note that the complement of a bull is a bull itself, and similarly for C_5. The *diamond* is $K_4 - e$, i.e., a four vertex clique minus one edge. $2K_2$ is the complement of C_4, i.e., it consists of four vertices a, b, c, d and two edges ab and cd.

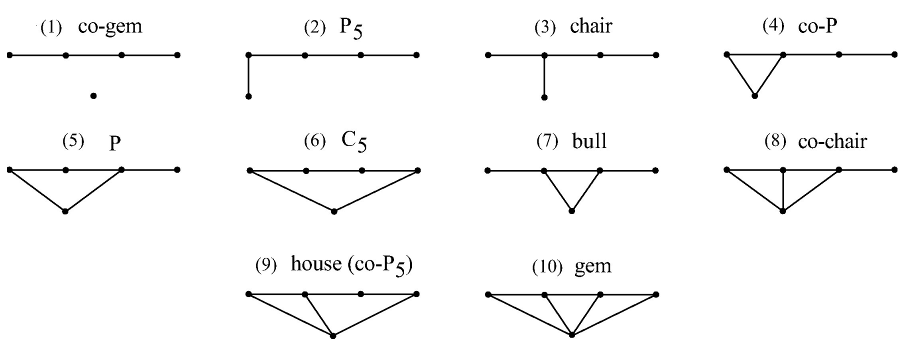

Fig. 2. All one-vertex extensions of a P_4.

A graph is a *split graph* if G is partitionable into a clique and a stable set.

Lemma 1 [21]. *G is a split graph if and only if G is* $(2K_2, C_4, C_5)$*-free.*

In what follows, we need the following classes of graphs:

- G is a *thin spider* if its vertex set is partitionable into a clique C and a stable set S with $|C| = |S|$ or $|C| = |S| + 1$ such that the edges between C and S are a matching and at most one vertex in C is not covered by the matching (an unmatched vertex is called the *head of the spider*). A thin spider with six vertices is also called a *net*.
- A graph is a *thick spider* if it is the complement of a thin spider.
- G is *matched co-bipartite* if its vertex set is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching.
- G is *co-matched bipartite* if G is the complement graph of a matched co-bipartite graph.
- A bipartite graph $B = (X, Y, E)$ is a *bipartite chain graph* [37], [34] if there is an ordering $x_1, x_2, \ldots, x_k$ of all vertices in X such that $N(x_i) \subseteq N(x_j)$ for all $1 \leq i < j \leq k$. (Note that then the neighborhoods of the vertices from Y are also linearly ordered by set inclusion.) If, moreover, $|X| = |Y| = k$ and $N(x_i) = \{y_1, \ldots, y_i\}$ for all $1 \leq i \leq k$, then B is prime.
- G is a *co-bipartite chain graph* if it is the complement of a bipartite chain graph.
- G is an *enhanced co-bipartite chain graph* if it is partitionable into a co-bipartite chain graph with cliques C_1, C_2 and three additional vertices a, b, c (a and c optional) such that $N(a) = C_1 \cup C_2$, $N(b) = C_1$, and $N(c) = C_2$, and there are no other edges in G.
- G is an *enhanced bipartite chain graph* if it is the complement of an enhanced co-bipartite chain graph.

2. Cographs, Clique-Width and Logical Expressibility of Problems

The P_4-free graphs (also called *cographs*) play a fundamental role in graph decomposition; see [16] for linear time recognition of cographs, see [14]–[16] for more information on P_4-free graphs and see [8] for a survey on this graph class and related ones.

For a cograph G, either G or its complement is disconnected, and the *cotree* of G expresses how the graph is recursively generated from single vertices by repeatedly applying join and co-join operations.

Note that the cographs are those graphs whose modular decomposition tree contains only join and co-join nodes as internal nodes.

Based on the following operations on vertex-labeled graphs, namely

(i) creation of a vertex labeled by integer l,
(ii) disjoint union (i.e., co-join),
(iii) join between all vertices with label i and all vertices with label j for $i \neq j$, and
(iv) relabeling all vertices of label i by label j,

the notion of *clique-width* $cwd(G)$ of a graph G is defined in [17] as the minimum number of labels which are necessary to generate G by using these operations. Cographs are exactly the graphs whose clique-width is at most 2.

A *k-expression* for a graph G of clique-width k describes the recursive generation of G by repeatedly applying these operations using at most k different labels.

Proposition 1 [18], [19]. *The clique-width $cwd(G)$ of a graph G is the maximum of the clique-width of its prime subgraphs, and the clique-width of the complement graph $\overline{G}$ is at most* $2 \cdot cwd(G)$.

Recently, the concept of the clique-width of a graph attracted much attention since it gives a unified approach to the efficient solution of many problems on graph classes of bounded clique-width via the expressibility of the problems in terms of logical expressions.

In [18] it is shown that every problem definable in a certain kind of Monadic Second Order Logic, called *LinEMSOL*$(\tau_{1,L})$ in [18], is linear-time solvable on any graph class with bounded clique-width for which a k-expression can be constructed in linear time.

Roughly speaking, MSOL(τ_1) is Monadic Second Order Logic with quantification over subsets of vertices but not of edges; MSOL$(\tau_{1,L})$ is the restriction of MSOL(τ_1) with the addition of labels added to the vertices, and LinEMSOL$(\tau_{1,L})$ is the restriction of MSOL$(\tau_{1,L})$ which allows one to search for sets of vertices which are optimal with respect to some linear evaluation functions.

The problems Vertex Cover, Maximum Weight Stable Set, Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL$(\tau_{1,L})$ definable problems.

Theorem 1 [18]. *Let $\mathcal{C}$ be a class of graphs of clique-width at most k such that there is an $\mathcal{O}(f(|E|, |V|))$ algorithm, which, for each graph G in $\mathcal{C}$, constructs a k-expression defining it. Then for every LinEMSOL$(\tau_{1,L})$ problem on $\mathcal{C}$, there is an algorithm solving this problem in time $\mathcal{O}(f(|E|, |V|))$.*

As an application, it was shown in [18] that P_4-sparse graphs and some variants of them have bounded clique-width; a graph is P_4*-sparse* if no set of five vertices in G induces at least two distinct P_4's [27], [28], [31]. From the definition, it is obvious that a graph is P_4-sparse if and only if it contains no C_5, P_5, $\overline{P_5}$, P, $\overline{P}$, chair, co-chair (see Figure 2). See [12] for a systematic investigation of superclasses of P_4-sparse graphs.

In [27], it was shown that the prime P_4-sparse graphs are the spiders (which were called *turtles* in [27]), and by Propositions 1 and 2(iii), it follows that P_4-sparse graphs have bounded clique-width; corresponding 4-expressions can be obtained in linear time.

Proposition 2. *The clique-width is at most*

(i) 3 *for chordless paths as well as for their complements*,
(ii) 4 *for chordless cycles as well as for their complements*,
(iii) 3 *for thin spiders*, 4 *for thick spiders*,
(iv) 3 *for bipartite chain graphs*, 4 *for co-bipartite chain graphs*,

(v) 4 *for matched co-bipartite as well as for co-matched bipartite graphs*,
(vi) 4 *for enhanced bipartite chain graphs as well as for enhanced co-bipartite chain graphs*,

and corresponding k-expressions, $k \in \{3, 4\}$, *can be obtained in linear time*.

The proof of Proposition 2 is straightforward.

A graph G is *distance-hereditary* [30] if for every connected induced subgraph H of G, the distance between any two vertices in H is the same as their distance in G.

Theorem 2 [1]. *A graph is distance-hereditary if and only if it is house-, hole-, domino-, and gem-free*.

Note that by Theorem 2 and Lemma 1, a split graph is gem-free if and only if it is distance-hereditary. See also [8] for further information on distance-hereditary graphs. In [25] Golumbic and Rotics have shown that the clique-width of distance-hereditary graphs is at most 3 and corresponding 3-expressions can be determined in linear time.

Finally, observe that trivially, the clique-width of a graph with n vertices is at most n. The following lemma by Johansson gives a slightly sharper bound.

Lemma 2 [32]. *If G has n vertices then $cwd(G) \leq n - k$ as long as $2^k + 2k \leq n$*.

Thus, for instance, the clique-width of a graph with nine vertices is at most 7.

3. Classification Strategy

Let $\mathcal{F}(P_4)$ denote the set of the ten one-vertex extensions of the P_4 (see Figure 2). For $\mathcal{F}' \subseteq \mathcal{F}(P_4)$, there are 1024 classes of $\mathcal{F}'$-free graphs. The two extreme cases are:

(i) For $\mathcal{F}' = \emptyset$, the class of $\mathcal{F}'$-free graphs is the class of all graphs.
(ii) Assume that a graph has at least five vertices. Then it is a cograph if and only if it contains none of the ten possible one-vertex extensions of a P_4. Thus, for $\mathcal{F}' = \mathcal{F}(P_4)$ (and for graphs with at least five vertices), the class of $\mathcal{F}'$-free graphs is the class of cographs.

Our aim is to give a complete classification of all the other classes of $\mathcal{F}'$-free graphs with respect to their clique-width. To this purpose, in Figure 3 we denote graph classes by the numbers of their forbidden subgraphs as they are enumerated in Figure 2; for example, the (P_5,gem)-free graphs are the class denoted by (2,10), and the class of complements of these graphs is the class of (co-P_5,co-gem)-free graphs, i.e., the class denoted by (1,9). Recall that by Proposition 1, a class has bounded clique-width if and only if the class of its complement graphs has bounded clique-width. Thus, it suffices to mention the class with the lexicographically smaller name (in the example, (1,9) and not (2,10)).

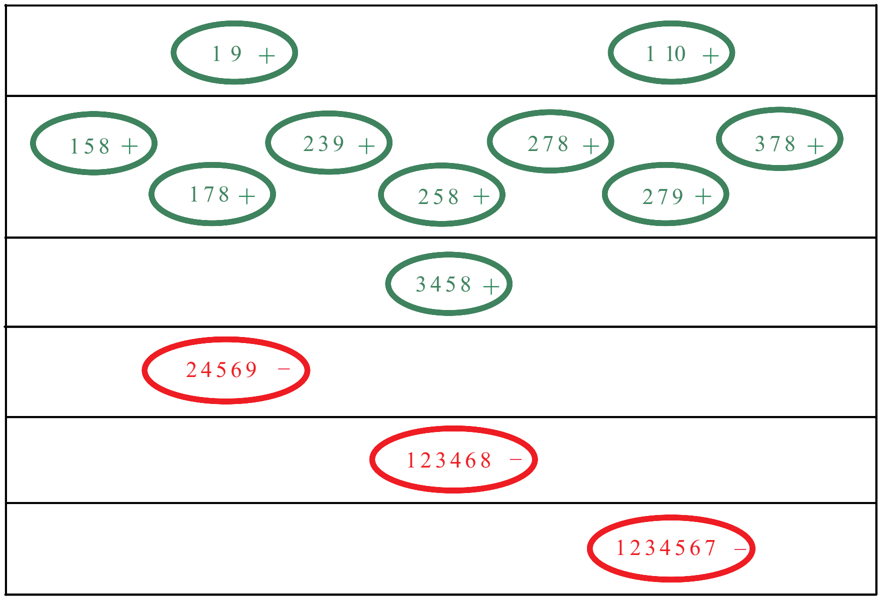

Fig. 3. Essential classes for all combinations of forbidden one-vertex P_4 extensions; $+$ $(-)$ denotes bounded (unbounded) clique-width.

Summarizing, Figure 3 describes all inclusion-minimal classes of unbounded clique-width and all inclusion-maximal classes of bounded clique-width (for best upper bounds see Proposition 3 at the end of Section 4). Note that any subclass of a class of bounded clique-width has bounded clique-width as well, whereas any superclass of a class of unbounded clique-width has unbounded clique-width as well. Thus, the classification is obtained by two types of results:

- bounded clique-width of some inclusion-maximal classes;
- unbounded clique-width of some inclusion-minimal classes

such that all the other classes will be a subclass of a class of bounded clique-width or a superclass of a class of unbounded clique-width.

3.1. *Unbounded Clique-Width*

All facts on unbounded clique-width used in this paper are based on the following results by Makowsky and Rotics [35]:

Theorem 3 [35].

(i) *The following grid types* (*see Figure* 4) *have unbounded clique-width*:
— *the* F_n *grid*;
— *the* $H_{n,q}$ *grid*.
(ii) *Split graphs have unbounded clique-width*.

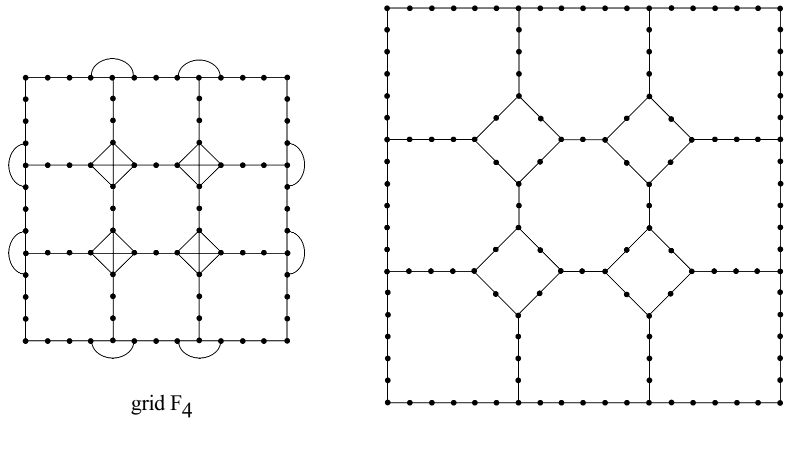

Fig. 4. The grids F_4 and $H_{4,2}$.

The F_n grid and the $H_{n,q}$ grid are constructed from an $n \times n$ square grid G_n as follows (see [35]): Let $n \geq 4$ and $q \geq 2$.

(i) F_n is the result of the following operations:
— Replace every edge of G_n by a simple path with four edges, introducing three new vertices which are the internal vertices of the path. Let G'_n denote the resulting graph.
— For all vertices v of degree 3 in G'_n do the following: for the two neighbors u and w of v such that u, v and w are in the same row or column in G'_n, add an edge connecting u to w. Let G''_n denote the resulting graph.
— For all vertices v of degree 4 in G''_n do the following: for the four neighbors u_1, u_2, u_3, u_4 of v, omit v from G''_n and add all edges connecting u_i and u_j, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, such that u_1, u_2, u_3, u_4 will form a clique. Then F_n is the resulting graph.

(ii) $H_{n,q}$ is the result of the following operations:
— Replace every edge of G_n by a simple path with three edges, introducing two new vertices which are the internal vertices of the path. Let G'_n denote the resulting graph.
— For all vertices v of degree 4 in G'_n do the following: for the four clockwise neighbors u_1, u_2, u_3, u_4 of v, omit v from G'_n and add the edges $u_i u_{i+1}$ (index arithmetic modulo 4) such that u_1, u_2, u_3, u_4 induce a C_4 in G'_n. Let G''_n denote the resulting graph.
— Replace every edge of G''_n by a simple path with q edges, introducing $q - 1$ new vertices which are the internal vertices of the path. Then $H_{n,q}$ is the resulting graph.

By the definitions of F_n and $H_{n,q}$, F_n grids do not contain chair, P, C_5, co-chair, house, or gem, and so their complements are (1,2,3,4,6,8)-free. Also, $H_{n,q}$ grids do not contain co-P, P, C_5, bull, co-chair, house, or gem, and so their complements are (1,2,3,4,5,6,7)-free. Moreover, by Lemma 1, split graphs do not contain P_5, co-P, P, C_5, or house, i.e., they are (2,4,5,6,9)-free. This proves the next lemma.

Lemma 3. *The following graph classes have unbounded clique-width*:

(i) (1,2,3,4,6,8)-*free graphs*;
(ii) (1,2,3,4,5,6,7)-*free graphs*;
(iii) (2,4,5,6,9)-*free graphs*.

This implies unbounded clique-width for all classes with the label "−" in Figures 3 and 7.

3.2. *Bounded Clique-Width*

Proposition 1 shows that in order to obtain an upper bound on the clique-width of a graph class, it suffices to study its prime graphs. This is done extensively here and in other papers, and Theorem 4 is a good example:

Theorem 4 [10]. *If G is a prime* (*diamond*,*co-diamond*)-*free graph then G or $\overline{G}$ is a matched co-bipartite graph or G has at most nine vertices*.

Using linear-time modular decomposition, this gives a linear-time construction for corresponding 9-expressions (the bound 9 can be slightly improved).

Other basic results on bounded clique-width used here are the subsequent Theorems 5–14:

Theorem 5 ((1,9)) [6]. *The clique-width of* (P_5,*gem*)-*free graphs is at most* 5.

The proof given in [6] is long and technically involved and does not provide a linear-time construction of corresponding 5-expressions.

Theorem 6 ((1,10)) [7]. *The clique-width of* (*gem*,*co-gem*)-*free graphs is at most* 16.

The proof of Theorem 6 is technically very involved and does not give linear-time algorithms for constructing corresponding 16-expressions.

Theorem 7 ((1,5,8)) [9]. *If G is a prime* (*chair*,*co-P*,*gem*)-*free graph then G fulfills one of the following conditions*:

(i) *G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$*;
(ii) *G is a thin spider*;
(iii) *G is a co-matched bipartite graph*;
(iv) *G has at most nine vertices*.

Theorem 8 ((2,3,9)) [23]. *If G is a prime* (P_5,$\overline{P_5}$,*chair*)-*free graph then G is either a co-bipartite chain graph or a spider or C_5*.

Theorem 9 ((2,5,8)) [12]. *If G is a prime* (*chair,co-P,house*)*-free graph then G fulfills one of the following conditions*:

(i) *G is an induced path* P_k, $k \geq 4$, *or an induced cycle* C_k, $k \geq 5$;
(ii) *G is a co-matched bipartite graph*;
(iii) *G is a spider.*

Theorem 10 ((2,7,8)) [4]. *If G is a prime* ($\overline{P_5}$,*bull,chair*)*-free graph then G is either a co-matched bipartite graph or an induced path or cycle or* $\overline{G}$ *is* (P_5,*diamond*)*-free*.

This describes the structure of prime ($\overline{P_5}$,bull,chair)-free graphs completely since the following result about prime (P_5,diamond)-free graphs is known:

Theorem 11 [2]. *Prime* (P_5,*diamond*)*-free graphs are matched co-bipartite or a thin spider or an enhanced bipartite chain graph or have at most nine vertices*.

Theorem 12 ((2,7,9)) [22]. *If G is a prime* (P_5,$\overline{P_5}$,*bull*)*-free graph then G or* $\overline{G}$ *is a bipartite chain graph or a* C_5.

Theorem 13 ((3,7,8)) [4]. *If G is a prime* (*bull,chair,co-chair*)*-free graph then G or* $\overline{G}$ *is either a co-matched bipartite graph or an induced path or cycle*.

Theorem 14 ((3,4,5,8)) [12]. *If G is a prime* (*chair,co-chair,P,co-P*)*-free graph then one of the following conditions holds*:

(i) *G or* $\overline{G}$ *is an induced path* P_k, $k \geq 4$, *or an induced cycle* C_k, $k \geq 5$;
(ii) *G is a spider*;
(iii) *G has at most nine vertices*.

In Section 4 we derive structure results for some of the subclasses of (1,9) and (1,10), namely (1,2,9), (1,2,10), (1,4,10), (1,3,9), (1,7,9), (1,8,9), (1,3,10), (1,4,9), and we give a detailed structure description for the class (1,7,8) and a shorter proof for Theorem 7 on (1,5,8). These results imply linear-time algorithms for constructing corresponding k-expressions.

Finally, a straightforward case analysis gives the complete classification described in Figure 3.

3.3. *Further Tools*

In this subsection we collect some further helpful facts which are needed in Section 4.

Lemma 4 [29]. *If a prime graph contains an induced* C_4 (*induced* $2K_2$) *then it contains an induced co-*P_5 *or A or domino* (*induced* P_5 *or co-A or co-domino*).

The proof of Lemma 4 can be extended in a straightforward way to the case of a diamond instead of a C_4. For this purpose we call *d-A* the graph resulting from an *A* graph by

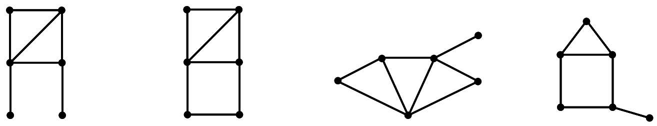

Fig. 5. The d-A and d-domino and their complements.

adding an additional diagonal edge in the C_4, and *d-domino* the graph resulting from a domino graph by adding an additional diagonal edge in one of the C_4's—see Figure 5.

Lemma 5. *If a prime graph contains an induced diamond* (*co-diamond*) *then it contains an induced gem or d-A or d-domino* (*co-gem or co-d-A or co-d-domino*).

See, e.g., [2] for a proof of Lemma 5.

Lemma 6 [5]. *Prime* (P_5,*gem*)*-free graphs containing a co-domino are matched co-bipartite*.

Lemma 7 [4]. *Prime chair-free bipartite graphs are either co-matched bipartite or an induced path or cycle*.

Lemma 8 [11]. *Prime chair-free split graphs are spiders*.

The following lemma is a useful result by De Simone:

Lemma 9 [20]. *Every prime graph is either bipartite or co-diamond-free or is an odd hole or contains an induced subgraph from the following family F of graphs* (*see Figure* 6 *for* $F_1, \ldots, F_6$):

1. *an odd hole plus a new vertex adjacent to exactly one of the hole vertices* (*called an* odd apple *in* [20]);
2. F_1: *thin spider with six vertices* (*i.e.*, *net*);
3. F_2: *a bull plus a new vertex adjacent to one of the bull vertices of degree* 1;
4. F_3: *the d-A*;
5. F_4: *the co-d-A*;
6. F_5: *the co-d-domino*;
7. F_6: *a house plus a new vertex adjacent to one of the house vertices of degree* 3.

Lemma 10 [4], [20]. *Prime* (*bull*,*chair*)*-free graphs containing a co-diamond are either co-matched bipartite or an induced path or cycle*.

Lemma 11. *Prime co-gem-free bipartite graphs are co-matched bipartite*.

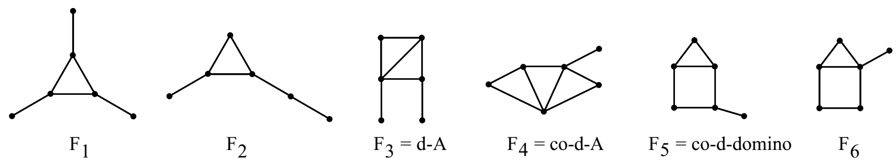

Fig. 6. Graphs F_1–F_6.

Proof. Let $B = (X, Y, E)$ be a prime co-gem-free bipartite graph. Assume first that B is co-diamond-free. If B is not co-matched then let $a \in X$, $b, c \in Y$, $b \neq c$, such that $ab \notin E$ and $ac \notin E$. Since B, as a prime graph, is connected, there is a vertex $d \in Y$ with $ad \in E$ but now a, b, c, d induce a co-diamond—contradiction. Assume now that the prime bipartite co-gem-free graph B contains a co-diamond. Then, by Lemma 5, it contains a co-d-A or co-d-domino but both of them are not bipartite—contradiction. □

Lemma 12. *Prime (co-diamond,gem)-free graphs containing a diamond have at most nine vertices.*

Proof. Let H be an induced subgraph of a graph G and assume that $V(H) = \{v_1, \ldots, v_h\}$. We use the following notation: For a set $S \subseteq \{1, \ldots, 5\}$, V_S denotes the set of all vertices $v \in G - H$ such that $N_H(v) = \{v_i \mid i \in S\}$. We also write V_{123} for $V_{\{1,2,3\}}$ and so on. Recall that a vertex not in H is a *k-vertex* for H if it has exactly k neighbors in H.

By Lemma 5 and since G is co-diamond- and gem-free, G contains a d-domino D. Let D consist of the P_5 Q with vertices 1, 2, 3, 4, 5 and edges 12, 23, 34, 45 and a 4-vertex a for Q being nonadjacent to 2 (thus, $a \in V_{1345}$).

Obviously, Q has no 0-vertex, no 1-vertex and no 5-vertex since G is co-diamond- and gem-free. The subsequent claims classify the other k-vertices for Q and describe their adjacencies.

Claim 3.1.

(i) *For all* 2*-vertices* v *for* Q, $v \in V_{24}$. *Moreover,* V_{24} *is a stable set.*
(ii) $V_{124} = V_{234} = V_{134} = V_{135} = V_{145} = V_{245} = V_{1234} = V_{2345} = V_{1245} = \emptyset$.

Proof. (i) Obviously, the only possible 2-vertices for Q are adjacent to 2 and 4 or to 1 and 5 but, in the last case, if x is a 2-vertex adjacent to 1 and 5 then $ax \in E$ holds since $a4, 2, x$ is no co-diamond, but then $345xa$ is a gem—contradiction. Thus, every 2-vertex for Q is adjacent to 2 and 4, and V_{24} is a stable set since G has no co-diamond.

(ii) If $x \in V_{124}$ then $1x, 3, 5$ is a co-diamond—contradiction, and analogously for $x \in V_{245}$. If $x \in V_{234}$ then $x3, 1, 5$ is a co-diamond—contradiction. If $x \in V_{134}$ then $x3a54$ is a gem if $ax \notin E$ and $1x45a$ is a gem otherwise—contradiction. If $x \in V_{135}$ then $a4, 2, x$ is a co-diamond if $ax \notin E$ and $1x34a$ is a gem otherwise—contradiction. If $x \in V_{145}$ then $3a5x4$ is a gem if $ax \notin E$ and $1x43a$ is a gem otherwise—contradiction. Obviously, $V_{1234} = V_{2345} = \emptyset$ since G is gem-free. If $x \in V_{1245}$ then $3a5x4$ is a gem if $ax \notin E$ and $1x43a$ is a gem otherwise—contradiction.

This shows Claim 3.1. □

Note that $V_{1345} \neq \emptyset$ since $a \in V_{1345}$.

Claim 3.2.

(i) $V_{1345} \textcircled{0} (V_{123} \cup V_{125} \cup V_{235} \cup V_{1235})$;
(ii) $V_{1235} \textcircled{0} V_{345}$ *and* $V_{1235} \textcircled{1} V_{123}$;
(iii) $V_{24} \textcircled{1} (V_{123} \cup V_{125} \cup V_{235} \cup V_{345})$;
(iv) $V_{123} \textcircled{1} (V_{125} \cup V_{235})$;
(v) $V_{345} \textcircled{0} (V_{123} \cup V_{125} \cup V_{235})$.

Proof. (i) Let $y \in V_{1345}$. If $x \in V_{123}$ then, since $x345y$ is no gem, $xy \notin E$. If $x \in V_{125}$ then, since $345xy$ is no gem, $xy \notin E$. If $x \in V_{235}$ or $x \in V_{1235}$ then, since $2xy43$ is no gem, $xy \notin E$.

(ii) If $x \in V_{1235}$ and $y \in V_{345}$ then, since $123yx$ is no gem, $xy \notin E$. If $x \in V_{1235}$ and $y \in V_{123}$ such that $xy \notin E$ then, by Claim 3.2(i), $ax \notin E$ and $ay \notin E$ and thus $a4, x, y$ is a co-diamond—contradiction.

(iii) Let $x \in V_{24}$. If $y \in V_{123}$ then, since $1y, x, 5$ is no co-diamond, $xy \in E$. If $y \in V_{125}$ then, since $1y, 3, x$ is no co-diamond, $xy \in E$. If $y \in V_{235}$ then, since $3y, 1, x$ is no co-diamond, $xy \in E$. If $y \in V_{345}$ then, since $5y, 1, x$ is no co-diamond, $xy \in E$.

(iv) Let $x \in V_{123}$. If $y \in V_{125}$ then, since $y1x32$ is no gem, $xy \in E$. If $y \in V_{235}$ then, since $1x, 4, y$ is no co-diamond, $xy \in E$.

(v) Let $x \in V_{345}$. If $y \in V_{123}$ then, since $123xy$ is no gem, $xy \notin E$. If $y \in V_{125}$ then, since $345yx$ is no gem, $xy \notin E$. If $y \in V_{235}$ then, since $23x5y$ is no gem, $xy \notin E$.

This shows Claim 3.2. □

Claim 3.3.

(i) $V_{123} = \emptyset$.
(ii) *If* $V_{125} \neq \emptyset$ *then* $V_{235} = V_{345} = \emptyset$.
(iii) *If* $V_{235} \neq \emptyset$ *then* $V_{345} = \emptyset$.
(iv) *If* $V_{1235} \neq \emptyset$ *then* $V_{125} = V_{235} = V_{345} = \emptyset$.

This means that at most one of the sets $V_{1235}, V_{125}, V_{235}, V_{345}$ is nonempty.

Proof of Claim 3.3. (i) By Claim 3.2, $V_{123} \cup \{2\}$ is a module, i.e., $V_{123} = \emptyset$ (because G is prime).

(ii) Let $V_{125} \neq \emptyset$ and $x \in V_{125}$. If $y \in V_{235}$ and if $xy \in E$ then $1xy32$ is a gem, and if $xy \notin E$ then $1x, 4, y$ is a co-diamond—contradiction. Thus, $V_{235} = \emptyset$.

If $y \in V_{345}$ and $z \in V_{1345}$ then, since $x2, y, z$ is no co-diamond, Claims 3.2(i) and (v) imply that $yz \in E$. Hence $V_{345} \textcircled{1} V_{1345}$, but then, by Claim 3.2, $V_{345} \cup \{4\}$ is a module, and thus $V_{345} = \emptyset$.

(iii) The proof that $V_{345} = \emptyset$ is completely analogous to the one in (ii).

(iv) Let $V_{1235} \neq \emptyset$ and $x \in V_{1235}$. If $y \in V_{125}$ then, since $y1x32$ is no gem, $xy \in E$ holds but now $32y5x$ is a gem—contradiction. If $y \in V_{235}$ then, since $1x3y2$ is no gem, $xy \in E$ holds but now $12y5x$ is a gem—contradiction.

Again, the proof that $V_{345} = \emptyset$ is completely analogous to the one in (ii) (except that this time Claim 3.2(ii) is used instead of Claim 3.2(v)).

This shows Claim 3.3. □

Claim 3.4.

(i) V_{1345} *is a module* (*i.e.*, $V_{1345} = \{a\}$).
(ii) V_{1235} *is a module*.
(iii) V_{345} *is a module*.
(iv) V_{125} *and* V_{235} *are modules*.
(v) $|V_{24}| \leq 2$.

Proof. (i) Let $x, y \in V_{1345}$ and $z \notin V_{1345}$ such that $xz \in E$ and $yz \notin E$. By Claim 3.2, $z \in V_{24}$ or $z \in V_{345}$.

If $z \in V_{24}$ then $xy \notin E$ since $1y4zx$ is no gem but now $zx5y4$ is a gem—contradiction. If $z \in V_{345}$ then $xy \in E$ since $xz, 2, y$ is no co-diamond but now $1y5zx$ is a gem—contradiction.

(ii) Let $x, y \in V_{1235}$ and $z \notin V_{1235}$ such that $xz \in E$ and $yz \notin E$. By Claim 3.3(iv), $V_{125} = V_{235} = \emptyset$, and thus by Claim 3.2, $z \in V_{24}$. Now $xy \in E$ since $zx1y2$ is no gem but now $z2y5x$ is a gem—contradiction.

(iii) By Claim 3.2, only V_{1345} can distinguish V_{345} vertices, and by Claim 3.4(i), $V_{1345} = \{a\}$. Assume that $x, y \in V_{345}$ and $ax \in E$, $ay \notin E$. Note that V_{345} is a clique since otherwise, for some $u, v \in V_{345}$ with $uv \notin E$, the vertices $12, u, v$ induce a co-diamond. Now by Claim 3.2, $\{x, 4\}$ is a homogeneous set—contradiction.

(iv) By Claim 3.2, the vertices in V_{125} and V_{235} can only be distinguished by each other and by vertices from V_{1235} but by Claim 3.3(iv), we may assume that $V_{1235} = \emptyset$, and by Claims 3.3(ii) and (iii), one of the sets V_{125} and V_{235} must be empty. Thus, the other one is also a module.

(v) By Claim 3.1, V_{24} is a stable set, and so, if vertex a has at least two nonneighbors u, v in V_{24} then $1a, u, v$ is a co-diamond; thus a has at most one nonneighbor in V_{24}. Likewise, $b \in V_{1235}$ has at most one nonneighbor in V_{24}. Moreover, by Claim 3.2, only vertices from V_{1345} and from V_{1235} can distinguish V_{24} vertices.

First assume that V_{1235} is nonempty; let $V_{1235} = \{b\}$. If V_{24} contains three vertices x, y, z then at least one of them, say x, is adjacent to a and b but then $\{3, x\}$ is a homogeneous set—contradiction.

Now assume that $V_{1235} = \emptyset$. Then, by Claim 3.2, only vertex $a \in V_{1345}$ can distinguish vertices from V_{24} and thus again $|V_{24}| \leq 2$ since otherwise two vertices in V_{24} (being either both adjacent to a or both nonadjacent to a) are a homogeneous set—contradiction.

This shows Claim 3.4. □

By Claims 3.1(ii), 3.3 and 3.4, it follows that G has at most nine vertices. This shows Lemma 12. □

1 2 3 −	1 2 4 −	1 2 5 −	1 2 6 −	1 2 7 −	1 2 8 −	1 2 9 +	1 2 10 +
1 3 4 −	1 3 5 −	1 3 6 −	1 3 7 −	1 3 8 −	1 3 9 +	1 3 10 +	1 4 5 −
1 4 6 −	1 4 7 −	1 4 8 −	1 4 9 +	1 4 10 +	1 5 6 −	1 5 7 −	1 5 8 +
1 5 9 +	1 6 7 −	1 6 8 −	1 6 9 +	1 6 10 +	1 7 8 +	1 7 9 +	1 7 10 +
1 8 9 +	2 3 4 −	2 3 5 −	2 3 6 −	2 3 7 −	2 3 8 −	2 3 9 +	2 4 5 −
2 4 6 −	2 4 7 −	2 4 8 −	2 4 9 −	2 5 6 −	2 5 7 −	2 5 8 +	2 6 7 −
2 6 8 −	2 6 9 −	2 7 8 +	2 7 9 +	3 4 5 −	3 4 6 −	3 4 7 −	3 4 8 −
3 5 6 −	3 5 7 −	3 6 7 −	3 6 8 −	3 7 8 +	4 5 6 −	4 5 7 −	4 6 7 −

Fig. 7. All combinations of three forbidden P_4 extensions; $+$ $(-)$ denotes bounded (unbounded) clique-width.

4. Structure and Clique-Width Results

Figure 7 contains all combinations of three forbidden P_4 extensions (enumeration according to Figure 2 similarly as in Figure 3). Again for complementary pairs of classes such as (1,2,9) and (2,9,10), we take only the lexicographically smaller one, namely (1,2,9). For best upper bounds on clique-width see Proposition 3.

Theorem 15 ((1,2,9)). *If G is a prime (P_5,co-P_5,gem)-free graph then G is distance-hereditary or a C_5.*

Proof. Let G be a prime (P_5,co-P_5,gem)-free graph. Assume first that G contains a C_5 C with vertices $v_1, \ldots, v_5$ and edges $\{v_i, v_{i+1}\}$, $i \in \{1, \ldots, 5\}$ (index arithmetic modulo 5). Since G is P_5-, house- and gem-free, C has no k-vertex for $k \in \{1, 4, 5\}$, 3-vertices for C have consecutive neighbors in C, and 2-vertices for C have nonconsecutive neighbors in C. Moreover, the C_5 dominates G, i.e., C has no 0-vertex. Now let $M_C(v_i)$ denote the set of vertices being adjacent to v_{i-1} and v_{i+1} and possibly to v_i (index arithmetic modulo 5). It is easy to see that $M_C(v_i)$ and $M_C(v_{i+1})$ form a join, and $M_C(v_i)$ and $M_C(v_{i+2})$ form a co-join. Thus, these vertex sets are modules, and therefore, $G = C$.

Now let G be a (C_5,P_5,co-P_5,gem)-free graph. Then G is (house,hole,domino,gem)-free which means that, by Theorem 2, G is distance-hereditary. □

The subsequent Theorems 16 and 17 are a simple consequence of Lemma 5.

Theorem 16 ((1,2,10), (1,4,10)). *If G is a prime* (P_5*,gem,co-gem*)*-free or* (*gem*, *co-P*, *co-gem*)*-free graph then G is* (*diamond*,*co-diamond*)*-free*.

Theorem 17 ((1,3,9), (1,7,9), (1,8,9)). *If G is a prime* (P_5*,gem,co-chair*)*-free or* (P_5*, gem,bull*)*-free or* (P_5*,gem,chair*)*-free graph then G is* (P_5*,diamond*)*-free*.

Thus, the structure of the classes considered in Theorems 16 and 17 is described in Theorems 4 and 11, respectively.

Theorem 18 ((1,3,10)). *If G is a prime* (*co-gem,chair,gem*)*-free graph then G or* $\overline{G}$ *is a matched co-bipartite graph or G has at most nine vertices*.

Proof. By Lemma 9, every prime graph is either bipartite or co-diamond-free or is an odd hole or contains an induced subgraph from the family F of graphs described in Lemma 9. Since an odd apple as well as the graphs F_3, F_4, F_5 and F_6 contain a chair and F_1 and F_2 contain a co-gem, and since G is prime (co-gem, chair, gem)-free, it follows that G is F-free and contains no odd hole of length at least 7. Thus G is either a C_5 or bipartite or co-diamond-free.

If G is bipartite, then, by Lemma 11, G is co-matched bipartite. So we assume that G is (co-diamond,gem)-free. If G contains no diamond then, by Theorem 4, G or $\overline{G}$ is a matched co-bipartite graph or G has at most nine vertices and we are done. If G contains a diamond then by Lemma 12, G has at most nine vertices. □

Theorem 19 ((1,4,9)). *If G is a prime* (P_5*,P,gem*)*-free graph then G is matched co-bipartite or a gem-free split graph* (*i.e.*, *distance-hereditary split graph*) *or a* C_5.

Proof. Let G be a prime (P_5,P,gem)-free graph. If G contains a co-domino then we are done by Lemma 6. Thus, by Lemma 4, assume that from now on, G is $2K_2$-free. Furthermore, if G contains no house then, by Theorem 15, G is a distance-hereditary graph or a C_5. Furthermore, if G is distance-hereditary and $2K_2$-free then it is ($2K_2$, C_4, C_5)-free and thus a gem-free split graph by Lemma 1.

Now assume that G contains a house with vertices $v_1, \ldots, v_5$. We call the house vertices of degree 3 the *inner vertices*, the house vertex adjacent to both inner vertices the *top vertex* and the two other vertices the *bottom vertices* of the house. Assume that v_1 and v_5 are the bottom vertices, $v_2(v_4)$ is the inner vertex adjacent to $v_1(v_5)$ and v_3 is the top vertex.

Claim 4.1. *If H is a house in G and v is a vertex not in H but adjacent to H then only the following neighborhood types are possible*:

(1) *v is adjacent to the inner vertices*;
(2) *v is adjacent to the top and bottom vertices*;
(3) *v is nonadjacent to exactly one bottom vertex.*

Proof. By a straightforward case analysis, one can see that all other cases lead to a $2K_2$, P or gem. □

Claim 4.2. *Every house in G dominates G.*

Proof. If a vertex y at distance 2 of a house H would exist then a common neighbor x of H and y leads to a $2K_2$ if x has adjacency type (1) and to a P if x has adjacency type (2) or (3)—contradiction. □

Note that $v_2 \in V_{134}$ and $v_4 \in V_{235}$ (V_{134} and V_{235} are defined as in the proof of Lemma 12).

Claim 4.3. *V_{134} and V_{235} are modules in G* (*and thus have size* 1).

Proof. Without loss of generality, assume that V_{134} is no module, i.e., there are $x, y \in V_{134}$ and $z \notin V_{134}$ with $xz \in E$, $yz \notin E$. Note that V_{134} is a clique since G is P-free.

Since $zxv_1v_4v_5$ is no P, $zv_1 \in E$ or $zv_4 \in E$ or $zv_5 \in E$. If only one of zv_1, zv_4, zv_5 is an edge then $zv_1v_4v_5y$ is a P—contradiction. Thus, at least two of these pairs are edges.

Case 1: $zv_1 \in E, zv_4 \in E, zv_5 \notin E$. Then, since $v_1zv_4v_5v_3$ is no P, $zv_3 \in E$ follows but now $z \in V_{134}$—contradiction.

Case 2: $zv_1 \in E, zv_5 \in E, zv_4 \notin E$. Then $yxzv_5v_1$ is a gem—contradiction.

Case 3: $zv_4 \in E, zv_5 \in E, zv_1 \notin E$. Then $yxzv_5v_4$ is a gem—contradiction.

Case 4: $zv_1 \in E, zv_5 \in E$ and $zv_4 \in E$. Then $yxzv_5v_1$ is a gem—contradiction.

This proves Claim 4.3. □

Claims 4.1, 4.2 and 4.3 imply that every vertex outside $\{v_1, v_2, v_3, v_4, v_5\}$ is either adjacent to v_2 and v_4 and nonadjacent to v_1 and v_5 or vice versa. Let

$L := \{v \mid v \in V$ and v is adjacent to v_2 and v_4 and nonadjacent to v_1 and $v_5\}$.

$R := \{v \mid v \in V$ and v is nonadjacent to v_2 and v_4 and adjacent to v_1 and $v_5\}$.

Note that $v_3 \in L$.

Claim 4.4. *L and R are cliques.*

Proof. Assume that there are $x, x' \in L$ with $xx' \notin E$. Then consider the co-connected component $C(x, x')$ in L containing x and x'. Since this cannot be a module, there must be a vertex u in R distinguishing two nonadjacent vertices in $C(x, x')$. Without loss of generality, say, $ux \in E$ and $ux' \notin E$ but now xv_2v_1ux' is a P—contradiction. In a completely analogous way, one can show that R is a clique. □

Claim 4.5. *The edges between L and R form a matching.*

Proof. Assume that there are $x, x' \in L$ having a common neighbor y in R. Then there must be a vertex $y' \in R$ distinguishing x and x'; without loss of generality, say $x'y' \in E$ and $xy' \notin E$ but now $v_4xyy'x'$ is a gem—contradiction. In a completely analogous way, one can show that there are no $y, y' \in R$ having a common neighbor in L. □

Claims 4.4 and 4.5 show that G is a matched co-bipartite graph if G contains a house. Theorem 19 follows. □

Now we give a shorter proof of Theorem 7; recall that it claims the following:

If G is a prime (chair,co-P,gem)-free graph then G fulfills one of the following conditions:

(i) G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;
(ii) G is a thin spider;
(iii) G is a co-matched bipartite graph;
(iv) G has at most nine vertices.

Proof of Theorem 7. Let G be a prime (chair,co-P,gem)-free graph. Then, by Lemma 5, G is diamond-free. Now we apply Lemma 9; the only graph in F not containing a chair, co-P or gem is the net (i.e., the thin spider with six vertices). Since G is prime (chair,co-P,gem)-free, it follows that G is bipartite or co-diamond-free or is an odd hole or G contains an induced net.

If G is bipartite then we are done by Lemma 7. If G is co-diamond-free then, by Theorem 4, G or $\overline{G}$ is a matched co-bipartite graph or G has at most nine vertices. A matched co-bipartite graph with at least eight vertices, however, contains a co-P, and thus, G is a co-matched bipartite graph or has at most nine vertices.

It remains to consider the case when G contains a net H, say with vertices v_1, v_2, v_3, v_4, v_5, v_6 and edges v_1v_2, v_2v_3, v_3v_1, v_1v_4, v_2v_5 and v_3v_6. We are going to show that in this case, G is a thin spider.

Using the fact that G is (chair,co-P,diamond)-free it is a matter of routine to see that only the following sets of k-vertices for H can be nonempty:

- the set V_{123} of 3-vertices for H which are adjacent to v_1, v_2 and v_3;
- the set V_0 of 0-vertices for H.

Since G is diamond-free,

$$V_{123} \text{ induces a clique.} \tag{1}$$

Since G is (co-P,chair)-free,

each $v \in V_{123}$ has at most one neighbor $w \in V_0$. (2)

Since G is (co-P)-free and G is connected, it follows from (2) that

V_0 is a stable set. (3)

By (1) and (3), G is a (diamond,chair)-free split graph and thus, by Lemma 8 and the fact that thick spiders contain a diamond, G is a thin spider. □

Theorem 20 ((1,7,8)). *If G is a prime (bull,chair,gem)-free graph then G fulfills one of the following conditions*:

(i) *G or $\overline{G}$ is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$*;
(ii) *G or $\overline{G}$ is a co-matched bipartite graph*;
(iii) *G has at most nine vertices*.

Proof. By Lemma 10, we may assume that G is co-diamond-free. If G is diamond-free then we are done by Theorem 4. If G contains a diamond then, by Lemma 12, G has at most nine vertices. □

Corollary 1. *The following classes (and their complement classes) have bounded clique-width and corresponding k-expressions can be determined in linear time*: (1,2,9), (1,2,10), (1,3,9), (1,3,10), (1,4,9), (1,4,10), (1,5,8), (1,7,8), (1,7,9), (1,8,9), (2,3,9), (2,5,8), (2,7,8), (2,7,9), (3,4,5,8), (3,7,8). *Thus, every LinEMSOL($\tau_{1,L}$) definable problem is solvable in linear time on these graph classes*.

By Theorem 5, the classes (1,5,9) and (1,6,9) have bounded clique-width and, by Theorem 6, the classes (1,6,10) and (1,7,10) have bounded clique-width but their structure is more complicated than the previous examples, and, so far, we do not know any linear time algorithm for determining k-expressions for these graphs.

Our best upper bounds on clique-width are described in the following proposition which gives a list of classes together with the classes of their complement graphs.

Proposition 3.

- $cwd(1,9) \leq 10$; $cwd(2,10) \leq 5$ [6];
- $cwd(1,10) \leq 16$ (*self-complementary*) [7];
- $cwd(1,2,9) \leq 4$; $cwd(2,9,10) \leq 5$;
- $cwd(1,2,10) \leq 7$; $cwd(1,9,10) \leq 7$;
- $cwd(1,3,9) \leq 7$; $cwd(2,8,10) \leq 5$;
- $cwd(1,3,10) \leq 7$; $cwd(1,8,10) \leq 7$;
- $cwd(1,4,9) \leq 4$; $cwd(2,5,10) \leq 5$;
- $cwd(1,4,10) \leq 7$; $cwd(1,5,10) \leq 7$;
- $cwd(1,5,8) \leq 7$; $cwd(3,4,10) \leq 7$;
- $cwd(1,5,9) \leq 10$; $cwd(2,4,10) \leq 5$;

- *cwd*(1,6,9)$\leq$ 10; *cwd*(2,6,10) $\leq$ 5;
- *cwd*(1,6,10)$\leq$ 16 (*self-complementary*);
- *cwd*(1,7,8)$\leq$ 7; *cwd*(3,7,10) $\leq$ 7;
- *cwd*(1,7,9)$\leq$ 7; *cwd*(2,7,10) $\leq$ 5;
- *cwd*(1,7,10)$\leq$ 16 (*self-complementary*);
- *cwd*(1,8,9)$\leq$ 7; *cwd*(2,3,10) $\leq$ 5;
- *cwd*(2,3,9)$\leq$ 4; *cwd*(2,8,9) $\leq$ 4;
- *cwd*(2,5,8)$\leq$ 4; *cwd*(3,4,9) $\leq$ 4;
- *cwd*(2,7,8)$\leq$ 7; *cwd*(3,7,9) $\leq$ 7;
- *cwd*(2,7,9)$\leq$ 4 (*self-complementary*);
- *cwd*(3,7,8)$\leq$ 4 (*self-complementary*);
- *cwd*(3,4,5,8)$\leq$ 7 (*self-complementary*).

The proof of items 3–22 of Proposition 3 is based on the structure results of this paper, on the fact that the clique-width of (P_5,diamond)-free graphs as well as of their complement graphs is at most 7, the clique-width of (P_5,gem)-free graphs is at most 5 (and thus the clique-width of their complement graphs is at most 10), the clique-width of (gem,co-gem)-free graphs is at most 16, as well as Lemma 2 for $n = 9$. Since sometimes the clique-width bound 7 is due to the fact that in one of the subcases, the graph has at most nine vertices, one can expect that a more detailed analysis of the particular graphs having at most nine vertices will help to improve the upper bound 7.

It seems to be extremely hard to show lower bounds for clique-width in general. In some exceptional cases, however, it is clear that the bound cannot be improved. Thus, C_5 has clique-width 4 since it can be constructed with four labels, and three labels do not suffice.

5. Conclusion

As before, let $\mathcal{F}(P_4)$ denote the set of the ten one-vertex extensions of the P_4 (see Figure 2). For $\mathcal{F}' \subseteq \mathcal{F}(P_4)$, there are 1024 classes of $\mathcal{F}'$-free graphs. Recall that Figure 3 shows all inclusion-minimal classes of unbounded clique-width and all inclusion-maximal classes of bounded clique-width. This classification is obtained from the previous results by a straightforward case analysis which shows the following (note that again we consider pairs of graph classes, namely a class together with the class of its complement graphs):

For $|\mathcal{F}'| \in \{8, 9\}$, all classes of $\mathcal{F}'$-free graphs have bounded clique-width. For $|\mathcal{F}'| = 7$, (1,2,3,4,5,6,7) is the only inclusion-minimal class of unbounded clique-width (enumeration with respect to Figure 2) whereas all the other classes (as subclasses of some class of bounded clique-width) have bounded clique-width, and similarly for $|\mathcal{F}'| = 6$ and $|\mathcal{F}'| = 5$. For $|\mathcal{F}'| = 4$ there is exactly one inclusion-maximal class of bounded clique-width, namely (3,4,5,8). For $|\mathcal{F}'| = 3$, the inclusion-maximal classes of bounded clique-width are (1,5,8), (1,7,8), (2,3,9), (2,5,8), (2,7,8), (2,7,9), (3,7,8). For $|\mathcal{F}'| = 2$, the only classes of bounded clique-width are (1,9) and (1,10), and for $|\mathcal{F}'| = 1$, all classes have unbounded clique-width.

Thus, we obtained a complete classification of graph classes defined by forbidden one-vertex extensions of the P_4 with respect to their clique-width. All these classes generalize cographs (which are the P_4-free graphs) having clique-width at most 2. In this way we find new important graph classes of bounded clique-width such as the classes mentioned in Corollary 1, and using linear-time modular decomposition, we obtain linear-time algorithms for constructing corresponding k-expressions. By the results of Courcelle et al. [18], this provides linear-time algorithms on these classes for all problems which can be formulated in a certain Monadic Second Order Logic called LinEMSOL($\tau_{1,L}$) in [18]. This improves some recently published results of other authors:

(i) Theorem 7 implies that every LinEMSOL($\tau_{1,L}$) definable problem can be solved in linear time on (chair,co-P,gem)-free graphs which drastically improves the results of [33] and [38] for the MWS problem on this class.
(ii) Theorem 19 implies that every LinEMSOL($\tau_{1,L}$) definable problem can be solved in linear time on (co-gem,co-P,house)-free graphs and on (P_5,P,gem)-free graphs which drastically improves the result of [26] for the MWS problem on this class.

The main open problems seem to be the following:

Question 1. Can one find a simple condition like planarity for minors and treewidth, that would give bounded clique-width if induced subgraph exclusion is considered?

This seems to be a very ambitious task but our results could eventually be a small step towards such a result. Moreover, our results could eventually help for solving the other big challenge on clique-width:

Question 2. What is the complexity of determining whether the input graph has clique-width at most k (for fixed k)?

This problem is solvable in polynomial time for $k = 3$ (see [13]) and open for $k \geq 4$.

Further Open Problems. Is there a linear time algorithm for determining a k-expression with constant k for the classes (1,9) and (1,10) (and the corresponding subclasses (1,5,9), (1,6,9), (1,6,10), (1,7,10))?

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