Eccentricity function in distance-hereditary graphs

Feodor F. Dragan *, Heather M. Guarnera

Algorithmic Research Laboratory, Department of Computer Science, Kent State University, Kent, OH, USA

ARTICLE INFO

ABSTRACT

A graph $G = (V, E)$ is distance hereditary if every induced path of G is a shortest path. In this paper, we show that the eccentricity function $e(v) = \max\{d(v, u) : u \in V\}$ in any distance-hereditary graph G is almost unimodal, that is, every vertex v with $e(v) > rad(G) + 1$ has a neighbor with smaller eccentricity. Here, $rad(G) = \min\{e(v) : v \in V\}$ is the radius of graph G. Moreover, we use this result to fully characterize the centers of distance-hereditary graphs. Several bounds on the eccentricity of a vertex with respect to its distance to the center of G or to the ends of a diametral path are established. Finally, we propose a new linear time algorithm to compute all eccentricities in a distance-hereditary graph.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

The eccentricity $e(v)$ of a vertex v is the length of a longest shortest path from v to any other vertex. In a distance-hereditary graph G, the length of any induced path between two vertices equals their distance in G [16]. The diameter $diam(G)$ (maximum eccentricity) and radius $rad(G)$ (minimum eccentricity) of distance-hereditary graphs have been extensively studied. A close relationship between diameter and radius was discovered in [9,24], where it was shown that $diam(G) \geq 2rad(G) - 2$. It was shown in [9] that with two sweeps of a Breadth-First Search (BFS) one can obtain a value that is very close to the diameter. In fact, any vertex v that is furthest from an arbitrary vertex u has eccentricity $e(v) \geq diam(G) - 2$. Later, Feodor Dragan and Falk Nicolai [14] showed that by using instead LexBFS (Lexicographic Breadth-First Search) one can get a vertex v (last visited by a LexBFS starting at any vertex u) with $e(v) \geq diam(G) - 1$, and additionally if $e(v)$ is even, then $e(v)$ exactly realizes the diameter of G. This yielded a linear time algorithm to compute the diameter as well as a diametral pair of vertices [9,14], i.e., a pair x, y such that $d(x, y) = diam(G)$. There is also a linear time algorithm to find a central vertex (a vertex with minimum eccentricity) and calculate the radius [9]. These results were very recently generalized in [5]; it follows from [5] that all vertex eccentricities of a distance-hereditary graph G can be computed in total linear time via a split decomposition of G.

Here, we establish further properties of the eccentricity function in distance-hereditary graphs. Understanding the eccentricity function and being able to efficiently compute the diameter, radius, and all vertex eccentricities is of great importance. For example, in the analysis of social networks (e.g., citation networks or recommendation networks), biological systems (e.g., protein interaction networks), computer networks (e.g., the Internet or peer-to-peer networks), transportation networks (e.g., public transportation or road networks), etc., the *eccentricity $e_G(v)$* of a vertex v is used to measure the importance of

---

* Corresponding author.
  E-mail addresses: dragan@cs.kent.edu (F.F. Dragan), hmichaud@kent.edu (H.M. Guarnera).

v in the network: the *eccentricity centrality index* of v [18] is defined as $\frac{1}{e_G(v)}$. A graph's eccentricity function is *unimodal* if every non-central vertex v has an adjacent vertex u with $e(u) < e(v)$. The unimodality of the eccentricity function has been studied in a variety of graph classes; for example, it is exactly unimodal in Helly graphs [8] and almost unimodal in (α_1, Δ)-metric graphs [13] (that includes all chordal graphs) and in hyperbolic graphs [1]. In particular, it was shown [13] that in a chordal graph G the unimodality of the eccentricity function can break for a (non-central) vertex v, i.e., all neighbors w of v satisfy $e(w) \geq e(v)$, only under very specific conditions: that $diam(G) = 2rad(G)$, that $e(v) = rad(G) + 1$, and that v is at distance 2 from a central vertex. We show in the main theorem of Section 3 that the same conditions hold for vertices of distance-hereditary graphs. This result, which is of independent interest, is a crucial intermediate step to establish the remaining results of this paper.

The center $C(G)$ (all vertices of G with minimum eccentricity and the graph induced by those vertices) of a distance-hereditary graph G is also of interest. Many graph classes have a well defined center. The center of a tree is either K_1 or K_2 [17], the center of a maximal outerplanar graph is one of seven special graphs [22], and more generally all possible centers of 2-trees are known [21]. Graph centers have also been characterized fully for chordal graphs [3]. In distance-hereditary graphs it is known [24] that the diameter of the center is no more than 3. This was later improved by Hong-Gwa Yeh and Gerard Chang [23] that either $diam(C(G)) = 3$ and $C(G)$ is connected or $C(G)$ is a cograph (which may not be connected), i.e., a P_4-free graph. Furthermore, any cograph is the center of some distance-hereditary graph. We complete the characterization of centers of distance-hereditary graphs by investigating the instance that $C(G)$ is not a cograph (i.e., $diam(C(G)) = 3$), in which case $C(G)$ takes the form of a graph H which is further described in Section 6, and moreover each such H is the center of some distance-hereditary graph. Finally, we obtain several bounds on the eccentricity of an arbitrary vertex v with respect to its distance to a mutually distant pair and also its distance to the center $C(G)$. A simple dynamic programming algorithm is also presented which computes all eccentricities in a distance-hereditary graph in total linear time by utilizing a pruning sequence which is a characteristic property of distance-hereditary graphs.

The main contributions of this paper are summarized as follows.

- We show that the eccentricity function in distance-hereditary graphs is almost unimodal: for every vertex $v \notin C(G)$ there is a neighbor w such that $e(w) < e(v)$ or $diam(G) = 2rad(G)$, $e(v) = rad(G) + 1$, and $d(v, C(G)) = 2$. We present several consequences of this result for obtaining the eccentricity of a vertex in a distance-hereditary graph G.
- We propose certificates for the diameter, the radius and all eccentricities and show that the eccentricity of any vertex is closely bounded by its distances to just two mutually distant vertices.
- We fully characterize the centers of distance-hereditary graphs. The center of a distance-hereditary graph G is either a cograph or takes the form of a graph H described in Section 6. Moreover, every cograph and each such H is the center of some distance-hereditary graph.
- We present a new linear time algorithm to compute all eccentricities in a distance-hereditary graph.

2. Preliminaries

Let $G = (V, E)$ be an undirected, simple (without loops or parallel edges), connected graph. Let $n = |V|$ and $m = |E|$. A *path* $P(v_0, v_k)$ is a sequence of vertices $v_0, ..., v_k$ such that $v_i v_{i+1} \in E$ for all $i \in [0, k-1]$; its *length* is k. A graph G is connected if there is a path between every pair of vertices. Let $d_G(x, y)$ be the distance between two vertices x and y in G, that is, the length of a shortest path from x to y. A subgraph H of a graph G is called *isometric* if the distance in H between any of its two vertices equals their distance in G. By $G - \{x\}$ we denote an induced subgraph of G obtained from G by removing a vertex $x \in V$. The *eccentricity* $e_G(v)$ of a vertex v is the maximum distance from v to any vertex. The subindex is omitted if G is known by context. The *diameter* $(diam(G))$ and *radius* $(rad(G))$ of a graph G is the maximum and minimum eccentricity of a vertex, respectively. The *center* is the set of vertices whose eccentricities are minimum: $C(G) = \{v \in V : e(v) = rad(G)\}$. It will be convenient to denote by $C(G)$ also the subgraph of G induced by set $C(G)$. We define $C^k(G) = \{v \in V : e(v) \leq rad(G) + k\}$. We denote the set of furthest vertices from v as $F(v) = \{u \in V : d(u, v) = e(v)\}$. Vertices x, y are considered to be *mutually distant* if $x \in F(y)$ and $y \in F(x)$; they are called *a pair of mutually distant vertices*. A pair $\{x, y\}$ is called *a diametral pair* if $d(x, y) = diam(G)$. The *interval* $I(x, y) = \{v \in V : d(x, v) + d(v, y) = d(x, y)\}$ is the set of all vertices that are on shortest paths between x and y. An interval *slice* is defined as $S_k(x, y) = \{v \in I(x, y) : d(x, v) = k\}$ for some non-negative integer k. We denote by $< S >$ the subgraph of G induced by the vertices $S \subset V$. Let also $d(v, S) = \min\{d(v, u) : u \in S\}$ and $diam(S) = \max\{d_G(x, y) : x, y \in S\}$.

The *neighborhood* of v consists of all vertices adjacent to v, denoted by $N(v)$, and the *closed neighborhood* of v is defined as $N[v] = N(v) \cup \{v\}$. The kth neighborhood of a vertex v is the set of all vertices of distance k to v, that is, $N^k(v) = \{u \in V : d(u, v) = k\}$. Whereas a *disk* of radius k centered at a set S (or a vertex) is the set of vertices of distance at most k to some vertex of S, that is, $D(S, k) = \{u \in V : d(u, S) \leq k\}$. A vertex v is said to be *universal* to a set S if $N(v) \supseteq S$. Let $V = \{v_1, ..., v_n\}$. For an n-tuple of non-negative integers $(r(v_1), ..., r(v_n))$, a subset $M \subseteq V$ is an *r-dominating set* for a set $S \subseteq V$ in G if and only if for every $v \in S$ there is a vertex $u \in M$ with $d(u, v) \leq r(v)$. We also say that M *r-dominates* S in G. If $r(v_i) = 1$ for all i, then we say that M *dominates* S in G. If $S = V$ then we say that M r-dominates G. Two vertex sets A and B of G are said to be *joined* if each vertex of A is adjacent to every vertex of B. A vertex is *pendant* if $|N(v)| = 1$. Two vertices v and u are *twins* if they have the same neighborhood or the same closed neighborhood. *True twins* are adjacent; *false twins* are not. A graph is distance-hereditary if and only if each of its connected induced subgraphs is isometric [16], that is, the length of any induced path between two vertices equals their distance in G. These graphs are also characterized
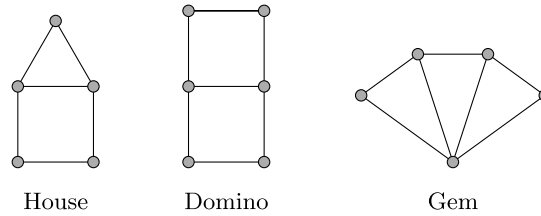
Fig. 1. Forbidden induced subgraphs in a distance-hereditary graph.

by a pruning sequence [2]: they can be dismantled by repeatedly removing either a pendant vertex or one vertex from a pair of twin vertices.

The following propositions provide basic information on distance-hereditary graphs necessary for the next sections.

Proposition 1. *[2,7] For a graph G, the following conditions are equivalent:*

 (i) *G is distance-hereditary;*

 (ii) *The house, domino, gem, and the cycles C_k of length $k \geq 5$ are not induced subgraphs of G (see Fig. 1);*

 (iii) *For an arbitrary vertex x of G and every pair of vertices $v, u \in N^k(x)$, that are connected in the same component of the graph $< V \setminus N^{k-1}(x) >$, we have $N(v) \cap N^{k-1}(x) = N(u) \cap N^{k-1}(x)$;*

 (iv) *(4-point condition) For any four vertices u, v, w, x of G at least two of the following distance sums are equal: $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(w, v)$. If the smaller sums are equal, then the largest one exceeds the smaller ones by at most 2.*

 (v) *G can be reduced to one vertex graph by a pruning sequence of one-vertex deletions: removing a pendant vertex or a single vertex from a pair of twin vertices.*

Proposition 2. *[9] Let G be a distance-hereditary graph with n-tuple $(r(v_1), ..., r(v_n))$ of non-negative integers and $M \subseteq V$. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v) + 1$ then M has an r-dominating clique C. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v)$ then there exists either a single vertex or a pair of adjacent vertices which r-dominates M.*

Proposition 3. *[9] For every vertex v of a distance-hereditary graph G, a furthest from v vertex $u \in F(v)$ satisfies $e(u) \geq 2rad(G) - 3$.*

Proposition 4. *Let G be a distance-hereditary graph and $x, y \in V$. Any vertex $v \in S_k(x, y)$ has $S_{k+1}(x, y) \subseteq N(v)$, i.e., neighboring interval slices are joined.*

Proof. Consider a vertex $u \in S_{k+1}(x, y) \cap N(v)$ and any other vertex $w \in S_{k+1}(x, y)$. Then u and w are connected in $< V \setminus N^{k+1}(x) >$ via shortest paths $P(u, y)$ and $P(w, y)$. By Proposition 1(iii), they share neighboring vertices in $S_k(x, y)$. Hence, $w \in N(v)$. □

Unless otherwise stated, the graph G is assumed to be distance-hereditary in all subsequent results.

3. Unimodality of the eccentricity function

Recall that the eccentricity function is unimodal in G if every non-central vertex v of G has a neighbor u such that $e(u) < e(v)$. Our main result of this section is that in distance-hereditary graphs any vertex with sufficiently large eccentricity does have a neighbor with strictly smaller eccentricity. Unimodality can break only at vertices with eccentricity equal to $rad(G) + 1$, but those vertices are close (within 2) to the center $C(G)$. Moreover, this can only occur when $diam(G) = 2rad(G)$. This property of the eccentricity function of distance-hereditary graphs aligns with known results for other graph classes, such as chordal graphs and the underlying graphs of 7-systolic complexes [13].

Our proof will be based on the following two lemmas.

Lemma 1. *If a vertex x of G has $e(x) = rad(G) + 1$, then $d(x, C(G)) \leq 2$.*

Proof. Let c be a central vertex closest to x. Consider any vertex $v \in S_1(c, x)$ and vertex $u \in F(v)$ furthest from v. As v is not central, $d(v, u) = rad(G) + 1$ and, by distance requirements, $d(c, u) = e(c) = rad(G)$. Hence, $u \in F(c) \cap F(v)$.

First we claim that $d(c, x) \leq 3$. Since $c \in I(v, u)$ and $v \in I(c, x)$, we have $d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2$. Consider the 4-point condition on vertices u, x, v, c. As two distance sums must be equal, then either $d(u, x) + d(c, v) = d(u, v) + d(c, x)$ or $d(u, x) + d(c, v) = d(u, c) + d(v, x)$. We have $d(u, x) + d(c, v) \leq e(x) + 1 = rad(G) + 2$. In the first case we get $d(u, x) + d(c, v) = d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2 = rad(G) + d(v, x) + 2$. Hence, $d(v, x) \leq 0$ and, by the triangle inequality, $d(x, C(G)) \leq 1$. In the second case we get $d(u, x) + d(c, v) = d(u, c) + d(v, x) = rad(G) + d(v, x)$. Hence, $d(v, x) \leq 2$ and, by the triangle inequality, $d(x, C(G)) \leq 3$, establishing the claim.

Assume now that $d(c, x) = 3$ and consider $y \in S_1(v, x)$. We next claim that $e(y) = rad(G) + 1$. By the choice of c, vertex y is non-central and so $e(y) \geq rad(G) + 1$. Since $y \in N(v) \cap N(x)$ with $e(v) = e(x) = rad(G) + 1$, by distance requirements, $e(y) \leq rad(G) + 2$. By way of contradiction assume that $e(y) = rad(G) + 2$. Consider a furthest vertex $y^* \in F(y)$. By distance requirements, $d(v, y^*) = d(x, y^*) = rad(G) + 1$ and $d(c, y^*) = rad(G)$. Since there is a (v, x)-path via vertex y in $< V \setminus N^{rad(G)}(y^*) >$, by Proposition 1(iii), the neighbors of v and x in $N^{rad(G)}(y^*)$ are shared. Therefore, $cx \in E$, contradicting with $d(c, x) = 3$. Thus, $e(y) = rad(G) + 1$ must hold.

We now obtain a general contradiction in two steps. Recall that $e(y) = e(v) = e(x) = rad(G) + 1$ and $e(c) = rad(G)$. First, consider the 4-point condition on vertices y, y^*, c, v. Consider three sums: $d(y, y^*) + d(c, v) = rad(G) + 2$, $d(y^*, c) + d(y, v) \leq rad(G) + 1$, and $d(y^*, v) + d(c, y) \leq rad(G) + 3$. Clearly, the first and the second sums are not equal. If the second and the third sums are equal, then $d(y^*, v) = d(y^*, c) + d(y, v) - d(c, y) \leq rad(G) - 1$, contradicting with $d(y^*, y) = rad(G) + 1$. Therefore, the first and the third sums are equal. Then $d(y^*, v) = d(y, y^*) + d(c, v) - d(c, y) = rad(G)$. Let $P(y^*, v)$ be any shortest path between y^* and v. Its length is $rad(G)$. Consider also the path $Q = P(y^*, v), y, x$ (extension of $P(y^*, v)$ that includes also y and x). In distance-hereditary graphs, every induced path is a shortest path. As $d(x, y^*) \leq e(x) = rad(G) + 1$, Q cannot be induced. As $d(y, y^*) = rad(G) + 1$, vertex x must be adjacent to some vertex on $P(y^*, v)$. To avoid large induced cycles C_k of length $k \geq 5$, x must be adjacent to a vertex $z \in P(y^*, v)$ which is a neighbor of v. Thus, $d(y^*, x) = rad(G)$. Necessarily, $zc \notin E$ since $d(x, c) = 3$. We also have that $d(y^*, c) \leq e(c) = rad(G)$.

Next, consider the 4-point condition on vertices y^*, c, x, v. We have $d(y^*, c) + d(x, v) \leq rad(G) + 2$, $d(y^*, v) + d(c, x) = rad(G) + 3$, and $d(y^*, x) + d(c, v) = rad(G) + 1$. Since at least two sums must be equal, necessarily $d(y^*, c) = d(y^*, x) + d(c, v) - d(x, v) = rad(G) - 1$. Now all distances from y^* are known. We have $d(c, y^*) = d(z, y^*) = rad(G) - 1$, $d(v, y^*) = d(x, y^*) = rad(G)$, and $d(y, y^*) = rad(G) + 1$. Since there is a (v, x)-path in $< V \setminus N^{rad(G)-1}(y^*) >$, by Proposition 1(iii), the neighbors of v and x in $N^{rad(G)-1}(y^*)$ are shared. Therefore, vertices x and c must be adjacent, contradicting with $d(x, c) = 3$.

Obtained contradiction proves the lemma. $\square$

Lemma 2. *If there is a non-central vertex v of G such that each vertex $w \in N(v)$ has $e(w) \geq e(v)$, then $e(v) = rad(G) + 1$ and $diam(G) = 2rad(G)$.*

Proof. Consider a vertex $u \in F(v)$, a vertex $x \in S_1(v, u)$ with minimal $|F(x)|$, and let $y \in F(x)$. By assumption, $e(x) \geq e(v)$ and v is non-central. If $d(v, y) = d(x, y) + 1$, then $e(v) \geq d(v, y) = d(x, y) + 1 = e(x) + 1 \geq e(v) + 1$, a contradiction. Thus, $d(x, y) - 1 \leq d(v, y) \leq d(x, y)$.

Consider the 4-point condition on vertices u, v, x, y. We have $d(u, v) + d(x, y) = e(v) + e(x) \geq 2rad(G) + 2$, $d(u, y) + d(x, v) \leq 2rad(G) + 1$, and $d(u, x) + d(v, y) = e(v) - 1 + d(v, y) \leq e(v) + e(x) - 1$. Since two sums must be equal, necessarily $d(u, y) + d(x, v) = d(u, x) + d(v, y)$. Hence $2rad(G) + 1 \geq d(u, y) + d(x, v) = d(u, x) + d(v, y) \geq (e(v) - 1) + (e(x) - 1) \geq 2e(v) - 2 \geq 2rad(G)$. Therefore, $2rad(G) \geq d(u, y) \geq 2rad(G) - 1$ and $e(v) \leq rad(G) + 3/2$. Since eccentricity is an integer and v is non-central, $e(v) = rad(G) + 1$ must hold.

It remains only to show that $diam(G) = 2rad(G)$. If $d(u, y) = 2rad(G)$, we are done. So, assume that $d(u, y) = 2rad(G) - 1$. We get $d(y, v) = d(u, y) + d(x, v) - d(u, x) = rad(G)$, and so $e(x) = rad(G) + 1$. Furthermore, $v \in I(x, y)$. The length of path $Q = P(u, v) \cup P(v, y)$ (the concatenation of $P(u, v)$ with $P(v, y)$) is $2rad(G) + 1$. As $d(u, y) = 2rad(G) - 1$, Q is not an induced path. Hence, there are vertices $s \in P(v, u)$ and $w \in P(v, y)$ such that $sw \in E$. To avoid large induced cycles C_k of length $k \geq 5$, necessarily $s \in S_1(x, u)$ and $w \in S_1(v, y)$ must hold. Then, w belongs to $S_1(v, u)$ as well as x. Since $y \in F(x) \setminus F(w)$ (note that $e(w) \geq e(v) = rad(G) + 1$ by assumption), by minimality of $|F(x)|$, there is a vertex $t \in F(w) \setminus F(x)$. Hence, $d(t, x) < rad(G) + 1$ and $d(t, w) = e(w) \geq rad(G) + 1$.

Now consider the 4-point condition on vertices x, y, w, t. We have $d(x, y) + d(w, t) = e(x) + e(w) \geq 2rad(G) + 2$, whereas $d(t, y) + d(w, x) \leq 2rad(G) + 2$ and $d(x, t) + d(w, y) \leq 2rad(G) - 1$. As $d(x, y) + d(w, t) - d(x, t) - d(w, y) \geq 3$, then only the two largest sums are equal: $d(x, y) + d(w, t) = d(t, y) + d(w, x)$. Hence, $diam(G) \geq d(t, y) = d(x, y) + d(w, t) - d(w, x) \geq 2rad(G)$. That is, $diam(G) = d(t, y) = 2rad(G)$. $\square$

We are ready to prove the main result of this section.

Theorem 1. *Every vertex $v \notin C(G)$ either has an adjacent vertex w with $e(w) < e(v)$ or $e(v) = rad(G) + 1$, $diam(G) = 2rad(G)$, and $d(v, C(G)) = 2$.*

Proof. If a non-central vertex v has no neighbors with smaller eccentricity then $d(v, C(G)) > 1$ and, by Lemma 2, $e(v) = rad(G) + 1$ and $diam(G) = 2rad(G)$. Thus, by Lemma 1, $d(v, C(G)) = 2$. $\square$

4. Certificates for eccentricities

We obtain as a consequence of Theorem 1 several new results for distance-hereditary graphs on lower and upper certificates for eccentricities, which were introduced in [12] as a way to compute exactly or approximately eccentricities in a

graph by maintaining upper and lower bounds. A set L (set U) of vertices is a *lower certificate* (respectively, an *upper certificate*) for eccentricities of G if it is used to obtain lower bounds (respectively, upper bounds) of eccentricities in G. Given all distances from a vertex v to all vertices in $L \cup U$ as well as the eccentricities of vertices in U, we have the following lower and upper bounds for the eccentricity of any vertex v [12]:

$$e_L(v) \le e(v) \le e^U(v), \text{ where } \begin{cases} e^U(v) = \min_{x \in U} d(v, x) + e(x), \\ e_L(v) = \max_{x \in L} d(v, x). \end{cases}$$

A lower certificate L (an upper certificate U) is said to be *tight* if $e_L(v) = e(v)$ ($e^U(v) = e(v)$, respectively) for all $v \in V$. A *diameter certificate* is a set U such that $e^U(v) \le diam(G)$ for all $v \in V$, and therefore the diameter is realized by $\max_{v \in V} e^U(v)$. A *radius certificate* is a set L such that $e_L(v) \ge rad(G)$ for all $v \in V$, and therefore the radius is realized by $\min_{v \in V} e_L(v)$. In what follows, we define the set of all diametral vertices of G as $D(G) = \{v \in V : e(v) = diam(G)\}$.

In this section we show that all eccentricities can exactly be determined in distance-hereditary graphs by computing distances from vertices of $C^1(G)$ to all vertices, since $C^1(G)$ forms a tight upper certificate. We also show that in distance-hereditary graphs the set $C(G)$ is a diameter certificate and the set $D(G)$ is a radius certificate (a kind of duality between $C(G)$ and $D(G)$). This agrees with radius and diameter certificates in chordal graphs [12] but, as we show later, this does not hold for arbitrary graphs.

We use the following corollary to Theorem 1.

Corollary 1. *Let G be a distance-hereditary graph.*
(i) *If $diam(G) < 2rad(G)$ then, for every pair of vertices $v \in V$ and $u \in F(v)$, there is a vertex $w \in I(v, u) \cap C(G)$ such that $u \in F(w)$.*
(ii) *For every pair of vertices $v \in V \setminus C(G)$ and $u \in F(v)$, there is a vertex $w \in I(v, u) \cap C^1(G)$ such that $u \in F(w)$.*

Proof. Consider any vertex $v \in V$ and $u \in F(v)$ and proceed by induction on $k := e(v)$. If $k = rad(G)$, then $w = v$ and we are done. If $k = rad(G) + 1$ and $diam(G) = 2rad(G)$ then again $w = v$ and we are done. If $k > rad(G) + 1$ or $k = rad(G) + 1$ and $diam(G) < 2rad(G)$ then, by Theorem 1, a neighbor z of v with $e(z) = k - 1$ satisfies $u \in F(z)$, and we can apply the induction hypothesis. $\square$

Lemma 3. *The set $C^1(G)$ is a tight upper certificate for all eccentricities of G.*

Proof. The statement follows from Corollary 1 and the definition of a tight upper certificate. $\square$

Lemma 4. *The center $C(G)$ is a diameter certificate of G.*

Proof. This is clear by Corollary 1 if $diam(G) < 2rad(G)$. Additionally, in any graph G with $diam(G) = 2rad(G)$ all central vertices $c \in C(G)$ and every diametral pair of vertices x, y satisfy $d(x, y) = d(x, c) + d(c, y) = d(x, c) + rad(G) = rad(G) + d(y, c) = 2rad(G)$. $\square$

Lemma 5. *The set $D(G)$ is a radius certificate of G.*

Proof. We first show that $D(G)$ is a radius certificate for any graph G if $diam(G) \ge 2rad(G) - 1$. If $D(G)$ is not a radius certificate, then there is a vertex $u \in V$ such that $max_{v \in D(G)} d(u, v) < rad(G)$. Thus, for any diametral pair x, y, $d(x, y) \le d(x, u) + d(u, y) \le (rad(G) - 1) + (rad(G) - 1) = 2rad(G) - 2$, a contradiction with $d(x, y) = diam(G) \ge 2rad(G) - 1$.

As in a distance-hereditary G, $diam(G) \ge 2rad(G) - 2$ holds [9,24], it remains to consider only the case when $diam(G) = 2rad(G) - 2$. Let S be the set of vertices u such that $d(u, t) \le rad(G) - 1$ for all $t \in D(G)$. By contradiction assume $D(G)$ is not a radius certificate and therefore S is not empty. Let $u \in S$ be a vertex which minimizes $|F(u)|$. Consider any diametral pair x, y and furthest from u vertex $v \in F(u)$. Necessarily $v \notin D(G)$ by the choice of u. Since $d(x, y) = 2rad(G) - 2$, $d(u, x) \le rad(G) - 1$ and $d(u, y) \le rad(G) - 1$, clearly $d(u, x) = d(u, y) = rad(G) - 1$.

Consider the 4-point condition on vertices v, u, x, y. We have that the largest distance sum is $d(v, u) + d(x, y) = d(v, u) + 2rad(G) - 2 \ge 3rad(G) - 2$, given that $d(v, x) + d(u, y) \le d(x, y) - 1 + rad(G) - 1 = 3rad(G) - 4$ and that $d(v, y) + d(u, x) \le d(x, y) - 1 + rad(G) - 1 = 3rad(G) - 4$. Therefore, the smaller sums are equal, establishing $d(v, x) = d(v, y)$. Moreover, since the difference between the largest sum and the other sums is at most 2, we get $d(v, u) = rad(G)$ and $d(v, x) = d(v, y) = 2rad(G) - 3$. So, $u \in C(G)$.

We claim that there is a vertex w such that $d(w, x) = rad(G) - 1$, $d(w, y) = rad(G) - 1$ and $v \notin F(w)$. Fix arbitrary shortest path $P(x, u)$, $P(y, u)$ and $P(u, v)$. Since $d(v, x) < d(v, u) + d(u, x) = diam(G) + 1$, path $Q = P(x, u) \cup P(u, v)$ is not induced. Hence, there must exist a chord between shortest path $P(x, u)$ and shortest path $P(u, v)$. Define vertices $t, w \in P(u, v)$, $s, z \in P(x, u)$, $q, p \in P(y, u)$, as shown in Fig. 2. Since $d(v, x) = 2rad(G) - 3$, we must have the chord $zt \in E$ or the chord $sw \in E$. By the same argument, there must exist a chord between shortest path $P(y, u)$ and shortest path $P(u, v)$ which is realized by chord $pt \in E$ or $qw \in E$. We note that if $zt \in E$ then $pt \notin E$ since $d(x, y) = 2rad(G) - 2$. Up to symmetry,
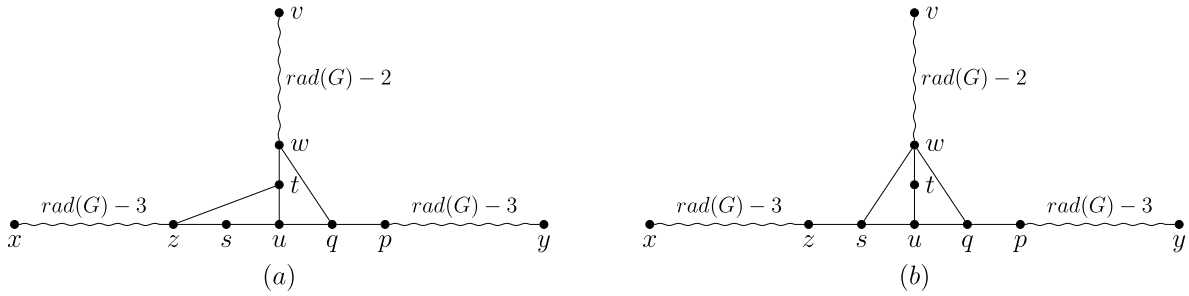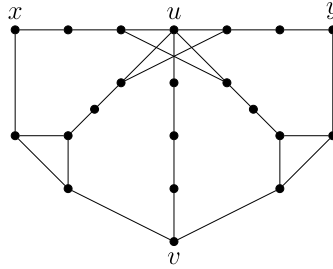
Fig. 2. Illustration to the proof of Lemma 5.

Fig. 3. A (non-distance-hereditary) graph G where $D(G)$ is not a radius certificate and $C(G)$ is not a diameter certificate.

we have two cases as shown in Fig. 2. In case (a) we have $zt, qw \in E$, and in case (b) we have $sw, qw \in E$. In either case vertex w satisfies the desired properties, establishing the claim.

We next claim that there is a vertex $\overline{w}$ such that $d(w, \overline{w}) \geq rad(G)$ and $d(u, \overline{w}) \leq rad(G) - 1$. On one hand, if $w \notin S$ then, by definition of S, there exists a vertex $\overline{w} \in D(G)$ such that $d(w, \overline{w}) \geq rad(G)$ and, by the choice of $u \in S$, we have $d(u, \overline{w}) \leq rad(G) - 1$. On the other hand, if $w \in S$ then, by minimality of $|F(u)|$ and since $v \notin F(w)$, there exists a vertex $\overline{w} \in F(w) \setminus F(u)$. As $\overline{w} \notin F(u)$, $d(u, \overline{w}) \leq rad(G) - 1$ and, as $\overline{w} \in F(w)$, $d(w, \overline{w}) \geq rad(G)$, establishing the claim.

Consider now the 4-point condition on vertices $v, u, w, \overline{w}$. Since $v \notin D(G)$ we have $d(v, \overline{w}) + d(w, u) \leq 2rad(G) - 3 + 2 = 2rad(G) - 1$. We also have $d(v, w) + d(\overline{w}, u) \leq rad(G) - 2 + rad(G) - 1 = 2rad(G) - 3$ and $d(v, u) + d(w, \overline{w}) \geq rad(G) + rad(G) = 2rad(G)$. Given that $d(v, u) + d(w, \overline{w})$ is strictly larger than the other sums, it must differ from them by at most 2. However, it differs by at least 3, giving a contradiction. □

As a consequence of these results, if the set $C^1(G)$ of a distance-hereditary graph G is known, then all vertex eccentricities in G can straightforwardly be found by performing a BFS from each vertex of $C^1(G)$. Similarly, if the set $C(G)$ ($D(G)$) is known, then the entire set $D(G)$ ($C(G)$, respectively) of G can be found. However, as we will discuss in Section 7, there is a more efficient approach to compute all eccentricities of a distance-hereditary graph.

We note that Lemma 4 and Lemma 5 do not hold for general graphs, as illustrated in Fig. 3 by a graph G with $diam(G) = 6$ and $rad(G) = 4$. Here $D(G) = \{x, y\}$ and $C(G) = \{u\}$, and all other vertices have eccentricity 5. However, $D(G)$ is not a radius certificate since $e_{D(G)}(u) = 3 < rad(G)$. Moreover, $C(G)$ is not a diameter certificate since $e^{C(G)}(v) = d(v, u) + e(u) = 8 > diam(G)$.

One aims also to minimize the size of a certificate. In trees, for example, a single diametral pair is a sufficient radius certificate rather than the full set of diametral vertices. Unfortunately this is not true for distance-hereditary graphs. The graph G in Fig. 4 illustrates that every diametral vertex is necessary to establish a radius certificate. Graph G consists of a clique of vertices $\{u_1, ..., u_\ell\}$ and a clique of vertices $\{v_1, ..., v_\ell\}$, where each u_i is adjacent to all vertices $v_{j \neq i}$, and each u_i and v_i has a pendant vertex x_i and y_i, respectively. G is distance-hereditary as it can be dismantled via a sequence of pendant and twin vertex eliminations. All vertices x_i and y_i are pendant, each u_i vertex is a false twin to v_i, and the remaining v_i vertices are true twins (as they form a clique in the remaining graph). Here $D(G)$ consists of all x_i and y_i vertices and $C(G)$ consists of all u_i and v_i vertices, where $diam(G) = d(x_i, y_i) = 4$ and $rad(G) = d(v_i, x_i) = d(u_i, y_i) = 3$. However, any $x_i \in D(G)$ has a vertex v_i such that $d(v_i, t) < rad(G)$ for all $t \in D(G) \setminus \{x_i\}$. By symmetry, the same is true for y_i and its counterpart u_i. Hence, all vertices of $D(G)$ are necessary to form a radius certificate. One can also show that all vertices of $C(G)$ are necessary to form a diameter certificate.

5. Eccentricities, mutually distant pairs and distances to the center

In this section, we show that the eccentricity of any vertex of a distance-hereditary graph is bounded by its distances to just two mutually distant vertices. Furthermore, the distance between any two mutually distant vertices is bounded by their
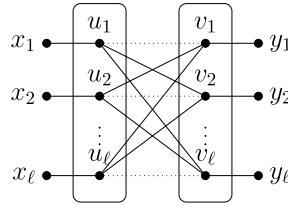
Fig. 4. A distance-hereditary graph G for which $D(G) \setminus \{t\}$ is not a radius certificate for any $t \in D(G)$ and for which $C(G) \setminus \{c\}$ is not a diameter certificate for any $c \in C(G)$.

distances to an arbitrary peripheral vertex (a vertex which is furthest for some other vertex). The unimodality behavior of the eccentricity function described in Theorem 1 gives also a relation between the eccentricity of a vertex and its distance to $C^1(G)$.

Lemma 6. *Let x, y be a mutually distant pair of G, and let $u \in V$ and $v \in F(u)$ be a furthest vertex from u. Let also $\alpha := d(u, x)$ and $\beta := d(u, y)$. Then,*

$$\max\{\alpha, \beta\} \leq e(u) \leq \max\{\max\{\alpha, \beta\}, \ \min\{\alpha, \beta\} + 2\} \leq \max\{\alpha, \beta\} + 2.$$

Moreover, if $e(u) = \max\{\alpha, \beta\} + 2$, then $\alpha = \beta = e(u) - 2$ and $d(v, x) = d(v, y) = d(x, y)$.

Proof. Let $v \in F(u)$. By the choice of v, we have $e(u) = d(u, v) \geq \max\{d(u, x), d(u, y)\}$. Consider the 4-point condition on vertices u, v, x, y. As x, y is a mutually distant pair, we have for the three distance sums that $d(u, v) + d(x, y) = e(u) + d(x, y)$, $d(u, x) + d(v, y) \leq d(u, x) + d(x, y) \leq e(u) + d(x, y)$, and $d(u, y) + d(v, x) \leq d(u, y) + d(x, y) \leq e(u) + d(x, y)$. Clearly the first sum is largest.

We first consider the case when the first sum equals one of the latter. Suppose that $d(u, v) + d(x, y) = d(u, x) + d(v, y)$. Then, $e(u) + d(x, y) = d(u, x) + d(v, y) \leq e(u) + d(x, y)$. Hence, $e(u) = d(u, x) = \max\{d(u, x), d(u, y)\}$. Suppose now that $d(u, v) + d(x, y) = d(u, y) + d(v, x)$. Then, $e(u) + d(x, y) = d(u, y) + d(v, x) \leq e(u) + d(x, y)$. Hence, $e(u) = d(u, y) = \max\{d(u, x), d(u, y)\}$. In either case, $e(u) = \max\{d(u, x), d(u, y)\}$.

We next consider the case when the two smaller sums are equal and differ from the largest one by at most 2. We have $e(u) = d(u, v) \leq d(v, y) + d(u, x) - d(x, y) + 2 = d(v, x) + d(u, y) - d(x, y) + 2$. Since $d(x, y)$ is not smaller than $d(v, y)$ and $d(v, x)$, we obtain $e(u) \leq d(u, x) + 2$ and $e(u) \leq d(u, y) + 2$, i.e., $e(u) \leq \min\{d(u, x), d(u, y)\} + 2$. Moreover, if $e(u) = \max\{d(u, x), d(u, y)\} + 2$, we must be in the latter case when the two smaller sums are equal (otherwise, $e(u) = \max\{d(u, x), d(u, y)\}$ as shown previously), and so $d(u, v) = e(u) = \min\{d(u, x), d(u, y)\} + 2$. Hence, $d(u, x) = d(u, y) = d(u, v) - 2$ and, since $d(u, x) + d(v, y) = d(u, y) + d(x, v)$, $d(v, y) = d(x, v)$ holds too. Combining this with the fact that $d(u, v) + d(x, y) - d(u, x) - d(y, v) \leq 2$, we obtain $d(x, y) \leq d(v, y)$ and, since x, y are mutually distant, necessarily $d(x, y) = d(v, y) = d(x, v)$, completing the proof. □

Lemma 7. *Let x, y be a mutually distant pair of G, and let $u \in V$ and $v \in F(u)$ be a furthest vertex from u. Let also $\alpha := d(v, x)$ and $\beta := d(v, y)$. Then,*

$$\max\{\alpha, \beta\} \leq d(x, y) \leq \max\{\max\{\alpha, \beta\}, \ \min\{\alpha, \beta\} + 2\} \leq \max\{\alpha, \beta\} + 2.$$

Moreover, if $d(x, y) = \max\{\alpha, \beta\} + 2$ then $\alpha = \beta = d(x, y) - 2$ and $d(u, x) = d(u, y) = d(u, v)$.

Proof. The proof is analogous to that of Lemma 6 and is omitted. The only difference is that now we argue from the perspective of $d(x, y)$ and not of $e(u)$. □

Fig. 5(a) illustrates that the upper bounds of Lemma 6 and Lemma 7 are sharp using vertices z and w for two opposing purposes. First, $e(z) = d(z, w) = \max\{d(x, z), d(y, z)\} + 2$, whereas $w \in F(z)$ has $d(x, y) = \max\{d(x, w), d(y, w)\}$. Secondly, $e(w) = d(z, w) = \max\{d(x, w), d(y, w)\}$, whereas $z \in F(w)$ has $d(x, y) = \max\{d(x, z), d(y, z)\} + 2$. Recall the implications of Lemma 6 and Lemma 7 which state that for a mutually distant pair x, y and fixed vertices $u \in V$ and $v \in F(u)$, if either $e(u)$ or $d(x, y)$ is realized by its upper bound as given in the above inequalities, then the other value is realized by its lower bound. So, it is not possible to obtain for the same $u \in V$ and $v \in F(u)$ that both $e(u) = \max\{d(x, u), d(y, u)\} + 2$ and $d(x, y) = \max\{d(x, v), d(y, v)\} + 2$ are true (Fig. 5(a) uses a different starting vertex u to illustrate both upper bounds - with $u := z$ and then with $u := w$). However, we show in Fig. 5(b) an example when for fixed vertices u and $v \in F(u)$, both $e(u) = \max\{d(x, u), d(y, u)\} + 1$ and $d(x, y) = \max\{d(x, v), d(y, v)\} + 1$ are true.

In the case when x and y form a diametral pair, Lemma 7 yields a result known from [23]. We have $diam(G) = d(x, y) \leq \max\{d(v, x), d(v, y)\} + 2 \leq e(v) + 2$.

Fig. 5. Illustration to the sharpness of Lemma 6 and Lemma 7.

Corollary 2. *[23] If vertex v of G is a furthest vertex from any $u \in V$, then $e(v) \geq diam(G) - 2$.*

Corollary 2 can be used to find a pair of mutually distant vertices of a distance-hereditary graph in linear time. Pick an arbitrary start vertex v_0. With at most five BFSs find vertices $v_1, \ldots, v_k$ ($2 \leq k \leq 5$) such that $v_i \in F(v_{i-1})$ and $d(v_k, v_{k-1}) = d(v_{k-1}, v_{k-2})$. Since, by Corollary 2, $e(v_1) \geq diam(G) - 2$, there are at most two improvements on $e(v_1)$ to get a required mutually distant pair v_{k-1}, v_{k-2}.

Now, by Lemma 6, the obtained distances from v_{k-1} and v_{k-2} to all vertices $u \in V$ already yield good lower bounds for vertex eccentricities in G (they are within 2 from exact eccentricities). We next show that, in some cases, they are even closer.

Corollary 3. *Let x, y be a mutually distant pair of G and let $u \in V$.*
(i) If $|d(x, u) - d(y, u)| \geq 2$, then $e(u) = \max\{d(x, u), d(y, u)\}$.
(ii) If $|d(x, u) - d(y, u)| = 1$ or $u \in C(G)$, then $\max\{d(x, u), d(y, u)\} \leq e(u) \leq \max\{d(x, u), d(y, u)\} + 1$.

Proof. By Lemma 6, $\max\{d(x, u), d(y, u)\} \leq e(u) \leq \max\{\max\{d(x, u), d(y, u)\}, \min\{d(x, u), d(y, u)\} + 2\}$. If $|d(x, u) - d(y, u)| \geq 2$, then $\max\{d(x, u), d(y, u)\} \geq \min\{d(x, u), d(y, u)\} + 2$ and therefore $e(u) = \max\{d(x, u), d(y, u)\}$. If $|d(x, u) - d(y, u)| = 1$, then $e(u) \leq \max\{d(x, u), d(y, u)\} + 1$. By contradiction assume now that $u \in C(G)$, $d(x, u) = d(y, u)$ and $e(u) = d(x, u) + 2$. By Proposition 3, we have $d(x, y) = e(x) \geq 2rad(G) - 3$. By the triangle inequality, $d(x, y) \leq d(x, u) + d(u, y) = 2(rad(G) - 2) = 2rad(G) - 4$, a contradiction. $\square$

In what follows, we analyze deeper the case when $d(u, x) = d(u, y)$. As the graph on Fig. 5 showed, in this case, $e(u) = \max\{d(x, u), d(y, u)\} + 2$ may happen. However, we demonstrate that it happens not very often. First we show that if $d(x, y)$ is odd, then still $e(u) \leq \max\{d(x, u), d(y, u)\} + 1$. For this we will need one auxiliary lemma.

Lemma 8. *Let u, x, y be vertices of G. If $d(u, x) = d(u, y)$ and $d(x, y) = 2k + 1$ for some integer k, then all vertices $s \in S_k(x, y)$ satisfy $d(u, x) = d(u, s) + k$.*

Proof. Let $s \in S_k(x, y)$ and $d(u, x) = d(u, y)$. Necessarily, $d(s, y) = k + 1$. Consider the 4-point condition on vertices x, y, u, s. We have $d(u, s) + d(x, y) = d(u, s) + 2k + 1$, $d(u, x) + d(s, y) = d(u, x) + k + 1$, and $d(u, y) + d(x, s) = d(u, x) + k$. Since at least two sums must be equal and the latter two sums are not, we consider the two remaining cases. If $d(u, s) + d(x, y) = d(u, x) + d(s, y)$, then $d(u, x) = d(u, s) + (2k + 1) - (k + 1) = d(u, s) + k$, and we are done. If $d(u, s) + d(x, y) = d(u, y) + d(x, s)$, then $d(u, x) = d(u, y) = d(u, s) + (2k + 1) - k = d(u, s) + k + 1$. However, by the triangle inequality, $d(u, x) \leq d(u, s) + d(s, x) = d(u, s) + k$, a contradiction. $\square$

Next lemma handles the case when $d(x, y)$ is odd.

Lemma 9. *Let x, y be a mutually distant pair of G. If $d(x, y)$ is odd, then any vertex $u \in V$ has $e(u) \leq max\{d(x, u), d(y, u)\} + 1$.*

Proof. Let $d(x, y) = 2k + 1$ for some integer k. By contradiction assume that $e(u) = \max\{d(x, u), d(y, u)\} + 2$. Consider a vertex $v \in F(u)$. By Lemma 6, when $e(u) = \max\{d(x, u), d(y, u)\} + 2$, we have $d(u, x) = d(u, y) = e(u) - 2$, and $d(x, y) = d(x, v) = d(y, v)$. Let $s \in S_k(x, y)$. By Lemma 8 applied to vertex u and to vertex v, we have $d(u, x) = d(u, s) + k$ and $d(v, x) = d(v, s) + k$. By the triangle inequality, $e(u) = d(u, v) \leq d(u, s) + d(s, v) = (d(u, x) - k) + (d(v, x) - k) = d(u, x) + d(v, x) - 2k = d(u, x) + d(x, y) - 2k = d(u, x) + 1$, contradicting with $e(u) = d(u, x) + 2$. $\square$

The following theorem summarizes the obtained bounds on the eccentricity of any vertex $u \in V$. The distances from u to a selected pair of vertices is very close to the eccentricity of u, and in some cases, measures it exactly.
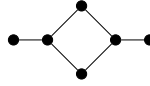
Fig. 6. A distance-hereditary graph in which every spanning tree is eccentricity 2-approximating.

Theorem 2. *Let x, y be mutually distant vertices of G. For every vertex $u \in V$, the following holds:*

$$\max\{d(x, u), d(y, u)\} \leq e(u) \leq \max\{d(x, u), d(y, u)\} + \begin{cases} 0, & \text{if } |d(x, u) - d(y, u)| \geq 2, \\ 1, & \text{if } |d(x, u) - d(y, u)| = 1 \text{ or } d(x, y) \text{ is odd}, \\ 2, & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 2, in distance-hereditary graphs, all eccentricities with an additive one-sided error of at most 2 can be computed in linear time. We further investigated the case in which $d(x, u) = d(y, u)$ and $d(x, y)$ is even, but this case proved to be difficult and did not lead to any improvements on the upper bounds for $e(u)$.

We remark that a 2-approximation of eccentricities is known for distance-hereditary graphs. One common approach to approximating eccentricities in a graph G is via an eccentricity k-approximating spanning tree T [4,10,13,20], i.e., a spanning tree T of G such that $e_T(v) - e_G(v) \leq k$ holds for each vertex v of G. Note that every additive tree k-spanner (a spanning tree T of G such that $d_T(x, y) \leq d_G(x, y) + k$ holds for every vertex pair x, y) is eccentricity k-approximating. However, there are graph families which do not admit any additive tree k-spanners and yet they have very good eccentricity approximating spanning trees. The introduction of eccentricity approximating spanning trees is an attempt to weaken the restriction of additive tree spanners and instead closely approximate only distances to most distant vertices, the eccentricities. This is fruitful especially for those graphs for which additive tree k-spanners with small k do not exist. For example, for every k there is a chordal graph without an additive tree k-spanner, though every chordal graph has an eccentricity 2-approximating spanning tree [13,20] computable in linear time [10]. More generally, all so-called δ-hyperbolic graphs (note that chordal graphs are 1-hyperbolic) have an eccentricity $(4\delta + 1)$-approximating spanning tree [4,11]. As distance-hereditary graphs are also 1-hyperbolic, that general result for δ-hyperbolic graphs implies that all distance-hereditary graphs have an eccentricity 5-approximating spanning tree. In fact, the situation with distance-hereditary graphs is even simpler. They have additive tree 2-spanners [19] (computable in linear time) and therefore eccentricity 2-approximating spanning trees. Furthermore, in general, the additive error 2 in an eccentricity approximating spanning tree cannot be improved. A distance-hereditary graph in Fig. 6 has no eccentricity 1-approximating spanning tree. Consider any edge uv of the inner C_4 which is not present in T; either $e_T(u) = e_G(u) + 2$ or $e_T(v) = e_G(v) + 2$. So, another approach is needed to efficiently compute all eccentricities in a distance-hereditary graph. In Section 7, we present a new linear time algorithm for that which utilizes a characteristic pruning sequence.

We turn now to a relation between the eccentricity of a vertex and its distance to $C(G)$ or $C^1(G)$.

Lemma 10. *Let $v \in V$ be any vertex of an arbitrary graph G and let k be an integer. If $e(v) = d(v, C^k(G)) + rad(G) + k$, then $e(v) = d(v, C^{k+1}(G)) + rad(G) + k + 1$.*

Proof. Suppose $e(v) = d(v, C^k(G)) + rad(G) + k$ and let $u \in F(v)$. Let $c \in C^k(G)$ be a closest vertex to v in $C^k(G)$. Consider an adjacent vertex $z \in S_1(c, v)$. By the choice of c, $e(z) = rad(G) + k + 1$ and therefore $z \in C^{k+1}(G)$. Then, $e(v) = d(v, c) + rad(G) + k = d(v, z) + d(z, c) + rad(G) + k \geq d(v, C^{k+1}(G)) + rad(G) + k + 1$. By the triangle inequality, also $e(c) \leq d(v, C^{k+1}(G)) + rad(G) + k + 1$. □

Lemma 11. *Let G be a distance-hereditary graph.*
(i) If $diam(G) < 2rad(G)$, then all vertices $v \in V$ satisfy $e(v) = d(v, C(G)) + rad(G)$.
(ii) All vertices $v \in V \setminus C(G)$ satisfy $e(v) = d(v, C^1(G)) + rad(G) + 1$.

Proof. The statements follow from Theorem 1 and Lemma 10 (see also Corollary 1). □

Lemma 12. *Let v be an arbitrary vertex of G and v' be an arbitrary vertex of $C(G)$ closest to v. Then, $d(v, C(G)) + rad(G) - 1 \leq e(v) \leq d(v, C(G)) + rad(G)$.*
Furthermore, all shortest paths $P \coloneqq (v' = x_0, x_1, ..., x_\ell = v)$, connecting v with v', satisfy the following:
(a) if $e(v) = d(v, C(G)) + rad(G)$, then $e(x_i) = d(x_i, C(G)) + rad(G) = i + rad(G)$ for each $i \in \{0, ..., \ell\}$;
(b) if $e(v) = d(v, C(G)) + rad(G) - 1$, then $e(x_i) = d(x_i, C(G)) - 1 + rad(G) = i - 1 + rad(G)$ for each $i \in \{3, ..., \ell\}$ and $e(x_1) = e(x_2) = rad(G) + 1$.

Proof. Let $r \coloneqq rad(G)$. By the triangle inequality, $e(v) \leq d(v, C(G)) + r$. Let $e(v) \geq r + 1$ and w be a vertex of $C^1(G)$ closest to v. By the triangle inequality and Lemma 1, $d(v, v') = d(v, C(G)) \leq d(v, w) + d(w, C(G)) \leq d(v, C^1(G)) + 2$. Combining with Lemma 11, one obtains $e(v) = d(v, C^1(G)) + r + 1 \geq d(v, C(G)) + r - 1$.

Consider an arbitrary shortest path $P := (v' = x_0, x_1, ..., x_\ell = v)$ connecting v with v'. As adjacent vertices can have eccentricities which differ by at most one, $e(v) = d(v, C(G)) + r$ implies $e(x_i) = d(x_i, C(G)) + r = i + r$ for each $i \in \{0, ..., \ell\}$. If $e(v) = d(v, C(G)) + r - 1$, there must exist an index $i \in \{1, ..., \ell - 1\}$ such that $e(x_i) = e(x_{i+1}) = r + i$ and $e(x_j) = r + j$ for $j \leq i$. We claim that $i = 1$ must hold.

We first establish that $i \leq 2$. By Theorem 1, there is a shortest path $Q^* := (w = u_2, ..., u_\rho = v)$ connecting v to a vertex $w \in C^1(G)$ closest to v, where $d(w, C(G)) \leq 2$ and $e(u_k) = r + k - 1$ for all $k \in 2, ..., \rho$. Let c be a vertex of $C(G)$ closest to w. We have that $d(v, C(G)) \leq d(v, c) \leq d(v, w) + d(w, c) \leq e(v) - e(w) + 2 = (d(v, C(G)) + r - 1) - (r + 1) + 2 = d(v, C(G))$. Therefore, vertex c is also a closest to v central vertex, $d(w, C(G)) = 2$, and $\rho = \ell$. Let now $Q := (c = u_0, u_1, u_2, ..., u_\ell = v)$ be a shortest path which joins shortest paths Q^* and any shortest path connecting u_2 to c. As c is a closest to v central vertex, necessarily $e(u_1) = e(u_2) = r + 1$. We claim that central vertices u_0 and x_0 are connected in $< V \setminus N^{d(v, C(G))-1}(v) >$ by respective paths to a vertex $t \in F(v)$. Let $u^* \in I(u_0, t)$, $u^* \neq u_0$. We have $d(u_0, t) \geq d(u^*, t) + 1$. Assume that $d(u^*, v) \leq d(v, C(G)) - 1$. Then, $e(v) = d(v, t) \leq d(v, u^*) + d(u^*, t) \leq (d(v, C(G)) - 1) + (d(u_0, t) - 1) \leq d(v, C(G)) + r - 2$, a contradiction with $e(v) = d(v, C(G)) + r - 1$. Therefore, any vertex u^* on a shortest (u_0, t)-path satisfies $d(v, u^*) \geq d(v, C(G))$ and, by symmetry, any vertex x^* on a shortest (x_0, t)-path also satisfies $d(v, x^*) \geq d(v, C(G))$. The combined paths connect vertices u_0 and x_0 in $< V \setminus N^{d(v, C(G))-1}(v) >$. As a result, for each $k \in \{0, ..., \ell - 1\}$, vertices u_k and x_k are connected in $< V \setminus N^{d(v, C(G))-k-1}(v) >$. Then, by Proposition 1(iii), $u_k x_{k+1} \in E$ and $u_{k+1} x_k \in E$. In particular, $u_2 x_3 \in E$, $u_2 x_1 \in E$, $u_1 x_2 \in E$, and $x_0 u_1 \in E$. Recall that $e(u_2) = r + 1$. If $i \geq 3$, then $e(x_3) = r + 3$; however, $e(x_3) \leq 1 + e(u_2) = r + 2$, a contradiction.

Assume now that $i = 2$. Hence, $e(x_2) = e(x_3) = r + 2$. Recall that $e(u_2) = e(u_1) = e(x_1) = r + 1$ and $e(x_0) = r$. Let $z \in F(x_2)$; then z is also furthest from vertices u_1, x_1, and x_0. Denote by Z a shortest path connecting x_0 and z, and let $z_0 \in Z$ be the vertex adjacent to x_0. By distance requirements from each vertex x_2, x_1, and u_1 to furthest vertex z, necessarily $z_0 x_2 \notin E$, $z_0 x_1 \notin E$, and $u_1 z_0 \notin E$. Since $d(x_3, x_0) = 3$, $z_0 x_3 \notin E$. As $d(u_2, z) \leq e(u_2) = r + 1$, the path obtained by joining shortest paths Z and (x_0, u_1, u_2) by their common end-vertex x_0 is not induced. The only possible chord is $u_2 z_0$. As u_2 and x_2 are connected in $< V \setminus N(x_0) >$, by Proposition 1(iii), $z_0 x_2 \in E$, a contradiction. □

Lemma 11 and Lemma 12 establish a close relationship between the eccentricity of a vertex and its distance to a closest vertex of $C(G)$ or of $C^1(G)$. We now use the following section to fully describe the structure of centers of distance-hereditary graphs.

6. Centers of distance-hereditary graphs

In this section, we investigate the structure of centers of distance-hereditary graphs and provide their full characterization. A subset $S \subseteq V$ is called m^3-convex if and only if S contains every induced path of length at least three between vertices of S. It is known from [15] that the centers of HHD-free graphs are m^3-convex. As every distance-hereditary graph is HHD-free, the centers of distance-hereditary graphs are m^3-convex, too. It is also known from [23] that $C(G)$ is either a cograph or a connected graph with $diam(C(G)) = 3$. As every connected subgraph of a distance-hereditary graph is isometric, when $diam(C(G)) = 3$, $C(G)$ is isometric and a distance-hereditary graph. We remark that if the diameter of a set S in a distance-hereditary graph is no more than 2 then, by definition, it induces a cograph (or, equivalently, a distance-hereditary graph of diameter at most 2).

To prove our main result of this section, we will need the following auxiliary lemmas.

Lemma 13. *Let x, y be a diametral pair of G, $diam(G) = 2rad(G)$, and $S := S_{rad(G)}(x, y)$. Then, S and $C(G)$ are cographs with $C(G) \subseteq S$, any vertex of $S_{rad(G)+1}(x, y) \cup S_{rad(G)-1}(x, y)$ is universal to S, and $C^1(G) \subseteq D(S, 1)$.*

Proof. Any central vertex $c \in C(G)$ has $d(x, c) \leq rad(G)$ and $d(y, c) \leq rad(G)$, therefore, by distance requirements, $C(G) \subseteq S$. By Proposition 4, any vertex $w \in S_{rad(G)+1}(x, y) \cup S_{rad(G)-1}(x, y)$ is universal to slice S and therefore universal to $C(G)$. Moreover, since the diameters of S and $C(G)$ are no more than 2, both are cographs.

Let now $c \in C^1(G)$ and by contradiction assume $c \notin D(S, 1)$. Necessarily $d(x, c) \leq rad(G) + 1$ and $d(y, c) \leq rad(G) + 1$. If $d(x, c) < rad(G)$ then, by distance requirements, $d(y, c) = rad(G) + 1$ and $c \in S_{rad(G)-1}(x, y)$. By Proposition 4, $c \in D(S, 1)$, a contradiction. If $d(x, c) = d(y, c) = rad(G)$ then $c \in S$, a contradiction. Hence, we can assume, without loss of generality, that $d(x, c) = rad(G) + 1$. Consider vertex $u \neq c$ on a shortest path $P(c, y)$ closest to x, and let $b \in S_{rad(G)+1}(x, y)$. Then, $d(y, u) \leq rad(G)$ as $d(y, c) \leq rad(G) + 1$. If $d(x, u) \geq rad(G) + 1$, then vertices b and c are connected in $< V \setminus N^{rad(G)}(x) >$ and therefore, by Proposition 1 (iii), share common neighbors in S. So, $c \in D(S, 1)$, a contradiction. If $d(x, u) \leq rad(G)$, then $d(x, u) = d(y, u) = d(y, c) - 1 = rad(G)$ as $d(x, y) = 2rad(G)$. Hence, $u \in S \cap N(c)$, a contradiction. □

Lemma 14. *Let x, y be a diametral pair of G, $diam(G) = 2rad(G) - 1$, and let $A := S_{rad(G)-1}(x, y)$ and $B := S_{rad(G)-1}(y, x)$. Then, $C(G)$ is a cograph and any edge $ab \in E$, where $a \in A$ and $b \in B$, satisfies $C(G) \subseteq D(\{a, b\}, 1)$. Moreover, there is a vertex $a \in A \cap C(G)$ and a vertex $b \in B \cap C(G)$.*

Proof. By Proposition 4, slices A and B are joined. Consider any $c \in C(G)$ and edge $ab \in E$ for any vertex pair $a \in A$ and $b \in B$. As $d(x, y) = 2rad(G) - 1$ and $e(c) = rad(G)$, $d(x, c) < rad(G)$ implies $d(x, c) = rad(G) - 1$ and $d(y, c) = rad(G)$. Hence, $c \in A$.

By symmetry, if $d(y, c) < rad(G)$ then $c \in B$. Assume now that $d(x, c) = d(y, c) = rad(G)$. Then, $b, c \in N^{rad(G)}(x)$. Consider vertex $u \neq c$ on a shortest path $P(c, y)$ closest to x. We have $d(y, u) \leq rad(G) - 1$ by the choice of u. If $d(x, u) \leq rad(G) - 1$, then $d(x, y) \leq d(x, u) + d(u, y) \leq 2rad(G) - 2$, a contradiction. Thus, $d(x, u) \geq rad(G)$. Vertices b and c are connected in $< V \setminus N^{rad(G)-1}(x) >$ by shortest paths to y. By Proposition 1 (iii), b and c share neighbors in A. Therefore, $a \in N(c)$ and, by symmetry, $b \in N(c)$. Hence, any central vertex either belongs to $A \cup B$ or is universal to $A \cup B$. Thus, $C(G) \subseteq D(\{a, b\}, 1)$. Additionally, since any pair of vertices in $C(G)$ is at most distance 2 apart, $C(G)$ is a cograph.

We now show the existence of vertices $a \in A$ and $b \in B$ such that $a, b \in C(G)$. Consider the family of disks $D(v, r(v))$ centered at each vertex v, where $r(v) = 1$ for all central vertices $v \in C(G)$ and $r(v) = rad(G) - 1$ for all others. Any two non-central vertices $u, v \in V \setminus C(G)$ have distance no more than the diameter, therefore $d(u, v) \leq 2rad(G) - 1 = r(u) + r(v) + 1$. Any two central vertices $u, v \in C(G)$ have distance no more than the diameter of the center, therefore $d(u, v) \leq 2 = r(u) + r(v)$. By definition, any central vertex $u \in C(G)$ sees any vertex $v \in V$ within $rad(G)$, and therefore $d(u, v) \leq rad(G) = r(u) + r(v)$. Hence, by Proposition 2, there is an r-dominating clique K. As any non-central vertex has distance $rad(G) - 1$ to a vertex of K, we have $K \subseteq C(G)$. Let $a \in K$ be closest to x and let $b \in K$ be closest to y. By distance requirements, ab must be an edge with $d(x, a) = rad(G) - 1$ and $d(b, y) = rad(G) - 1$. Therefore, $a \in A$ and $b \in B$. $\square$

Corollary 4. *If* $diam(G) \geq 2rad(G) - 1$ *then* $C(G)$ *is a cograph.*

It remains now to investigate the case when $diam(G) = 2rad(G) - 2$.

Lemma 15. *Let* $diam(G) = 2rad(G) - 2$, *and let* $M \subseteq C(G)$. *If all* $u, v \in M$ *satisfy* $d(u, v) = 2$, *then there is a vertex* $c \in C(G)$ *that is universal to* M.

Proof. Consider a disk of radius 1 centered at each $s \in M$ and a disk of radius $rad(G) - 1$ centered at each $v \in V \setminus M$. Any two vertices $u, v \in V \setminus M$ satisfy $d(u, v) \leq diam(G) = 2rad(G) - 2 = r(u) + r(v)$. Since $M \subseteq C(G)$, any $s \in M$ and $v \in V$ satisfy $d(s, v) \leq rad(G) = r(s) + r(v)$. By assumption, any two $s, t \in M$ satisfy $d(s, t) = 2 = r(s) + r(t)$. By Proposition 2, there is a single vertex or a pair of adjacent vertices r-dominating G. In the former case, we are done. Thus, consider the case when there is an r-dominating edge $ab \in E$. We have $a, b \in C(G)$ since all vertices $v \in V$ see some end-vertex of edge ab within $rad(G) - 1$. We claim that at least one end-vertex of edge ab is universal to M. By contradiction assume there exist vertices $u, v \in M$ which are adjacent to opposite ends of edge ab. Without loss of generality, let $u \in N(a) \setminus N(b)$ and $v \in N(b) \setminus N(a)$. Since $d(u, v) = 2$, we get in G either an induced C_5, or an induced house, or an induced gem. A contradiction obtained proves the lemma. $\square$

Lemma 16. *Let* x, y *be a diametral pair of* G, $diam(G) = 2rad(G) - 2$, *and let* $A := S_{rad(G)-2}(x, y)$, $S := S_{rad(G)-1}(x, y)$, *and* $B := S_{rad(G)-2}(y, x)$. *Then* $A \cup B \cup (S \cap C(G)) \subseteq C(G)$ *and there is a vertex* $c \in S \cap C(G)$. *Moreover,* $C(G) \subseteq D(S \cap C(G), 1)$.

Proof. Consider any $s \in A$. By Lemma 6, $e(s) \leq \max\{\max\{d(s, x), d(s, y)\}, \min\{d(s, x), d(s, y)\} + 2\} = d(s, y) = rad(G)$. Hence, $s \in C(G)$ and, by symmetry, $A \cup B \subseteq C(G)$.

By contradiction assume there is no central vertex in S. Let w be a vertex from S minimizing $|F(w)|$, and let $v \in F(w)$. Since $w \notin C(G)$, $d(w, v) \geq rad(G) + 1$. Denote by $s_1 \in S_1(w, x)$ and $s_2 \in S_1(w, y)$ two adjacent to w vertices on a shortest path from x to y. As previously established, both s_1 and s_2 are central since they belong to A and B, respectively. Thus, $d(s_1, v) \leq rad(G)$ and $d(s_2, v) \leq rad(G)$. Therefore, $d(s_1, v) = d(s_2, v) = rad(G)$ and $d(w, v) = rad(G) + 1$. Since $s_1, s_2 \in N^{rad(G)}(v)$ and are connected via w in the graph $< V \setminus N^{rad(G)-1}(v) >$, by Proposition 1(iii), there is a vertex $t \in N^{rad(G)-1}(v)$ adjacent to s_1 and s_2. As $t \in S_{rad(G)-1}(y, x)$ and $v \in F(w) \setminus F(t)$, by minimality of $|F(w)|$, there is a vertex $u \in F(t) \setminus F(w)$. By our assumption, $u \notin C(G)$, i.e., $d(u, t) \geq rad(G) + 1$. Consider the 4-point condition on vertices t, w, v, u. We have $d(v, u) + d(w, t) = d(v, u) + 2$, $d(v, t) + d(u, w) \leq rad(G) - 1 + rad(G)$, and $d(v, w) + d(u, t) \geq rad(G) + 1 + rad(G) + 1 = 2rad(G) + 2$. Since the latter two sums differ by more than 2, necessarily, $d(v, u) + d(w, t) = d(v, w) + d(u, t)$. Hence, $d(v, u) = d(v, w) + d(u, t) - d(w, t) \geq 2rad(G)$, a contradiction with $diam(G) = 2rad(G) - 2$. Thus, there is a vertex $c \in S \cap C(G)$.

Next, we establish an intermediate claim that $C(G) \subseteq D(M, 1)$, where $M := A \cup B \cup (S \cap C(G))$. By contradiction suppose there is a vertex $w \in C(G)$ with $w \notin D(M, 1)$. Consider arbitrary vertices $a \in A$ and $b \in B$. Thus, $d(a, b) = 2$, and $a, b \in M$ and, by the choice of w, necessarily $d(w, a) \geq 2$ and $d(w, b) \geq 2$. If $d(w, a) = d(w, b) = 2$ then, by Lemma 15 applied to the set $\{w, a, b\}$, there is a central vertex u adjacent to w, a, b. In this case $u \in S \cap C(G)$ and therefore $u \in M$, contradicting with $w \notin D(M, 1)$. Assume now, without loss of generality, that $d(w, a) \geq 3$. Consider the 4-point condition on vertices w, x, y, a. We have $d(x, y) + d(w, a) \geq 2rad(G) + 1$ is the largest sum since $d(x, w) + d(y, a) \leq 2rad(G)$ and $d(x, a) + d(w, y) \leq 2rad(G) - 2$. Since the smaller two sums must be equal and differ from the larger one by at most two, inequality $d(x, y) + d(w, a) \geq d(x, a) + d(y, w) + 3$ gives a contradiction which establishes the claim that $C(G) \subseteq D(M, 1)$.

Finally, we establish that $C(G) \subseteq D(S \cap C(G), 1)$. By contradiction assume there is a central vertex $w \in C(G) \setminus S$ which is not adjacent to any vertex of $S \cap C(G)$. By the previous claim, w is adjacent to some vertex from A or B. Without loss of generality, let $wa \in E$ for some vertex $a \in A$. Since $d(a, y) = rad(G)$, necessarily, $d(w, y) \geq rad(G) - 1$. If $d(w, y) = rad(G) - 1$ then $w \in S$, a contradiction. So, $d(w, y) = rad(G)$ must hold. Now, vertices w and a are connected in $< V \setminus N^{rad(G)-1}(y) >$. By Proposition 1 (iii), $N(w) \cap N^{rad(G)-1}(y) = N(a) \cap N^{rad(G)-1}(y)$. By Proposition 4, also $S \cap C(G) \subseteq N(a) \cap N^{rad(G)-1}(y)$. Thus, w is universal to $S \cap C(G)$, a contradiction. $\square$
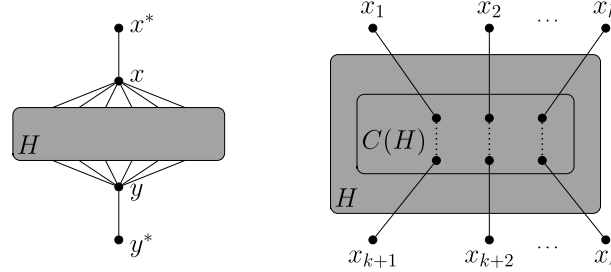
Fig. 7. Any cograph H (left), and any connected distance-hereditary graph H with diameter 3 where $C(H)$ is a connected cograph with radius 2 (right), is the center of some distance hereditary graph.

We are ready to prove the main result of this section.

Theorem 3. *Let H be a subgraph of a distance-hereditary graph G induced by $C(G)$. Either*
 (i) H is a cograph, or
(ii) H is a connected distance-hereditary graph with $diam(H) = 3$ and $C(H)$ is a connected cograph with $rad(C(H)) = 2$.
Furthermore, any such graph H is the center of some distance-hereditary graph.

Proof. If $diam(G) \geq 2rad(G) - 1$ then, by Lemma 13 and Lemma 14, H is a cograph. Assume now that $diam(G) = 2rad(G) - 2$ and $diam(H) = 3$ (if $diam(H) \leq 2$ then, by definition, H is a cograph). Then, H is a connected distance-hereditary graph [23], and so $2rad(H) - 2 \leq diam(H) \leq 2rad(H)$. On one hand, $rad(H) \geq \lceil (diam(H)/2) \rceil = 2$. On the other hand, $rad(H) \leq \lfloor (diam(H) + 2)/2 \rfloor = 2$. Hence, $rad(H) = 2$.

So, H is a connected distance-hereditary graph with $diam(H) = 3$ and $rad(H) = 2$. Consider the center $C(H)$ of H. First we show that $C(H)$ is connected. Let $r(u) = 1$ for each vertex $u \in H$. Then, any pair $u, v \in H$ satisfies $d_H(u, v) \leq diam(H) = r(u) + r(v) + 1$. By Proposition 2, there is a clique K in H dominating H. Since each vertex of K is at most distance $2 = rad(H)$ from every vertex of H, $K \subseteq C(H)$ holds. Moreover, every two vertices of $C(H)$ are connected through vertices of $K \subseteq C(H)$, implying that $C(H)$ is connected in H. In distance-hereditary graphs every connected subgraph is isometric. Hence, $C(H)$ is an isometric subgraph of H. As $rad(H) = 2$, every two vertices of $C(H)$ are at distance at most 2 from each other, implying $diam(C(H)) = 2$. Thus, $C(H)$ is a connected cograph with $rad(C(H)) \leq 2$.

We will show next that $rad(C(H)) = 2$, i.e., for any $c \in C(H)$ there is a vertex $z \in C(H)$ such that $cz \notin E$. Consider a vertex $t \in F(c)$ furthest from c in G. We have $d_G(c, t) = rad(G)$. Let $z \in H$ be a closest vertex to t which is central in G. Since $diam(G) = 2rad(G) - 2$, by Lemma 11, we have $d_G(t, z) = d_G(t, C(G)) = e_G(t) - rad(G) \leq diam(G) - rad(G) = rad(G) - 2$. Moreover, vertices z and c are not adjacent since $d_G(c, t) = rad(G)$ and $d_G(t, z) \leq rad(G) - 2$. But, since $c \in C(H)$, $d_G(c, z) = d_H(c, z) \leq 2$. Therefore, $d_H(c, z) = 2$ and $d_G(t, z) = rad(G) - 2$. We next establish that z belongs to $C(H)$. By contradiction, assume that there is a vertex $u \in H$ such that $d_H(z, u) > rad(H) = 2$. Then, $d_H(z, u) = diam(H) = 3$ and, by the choice of c ($c \in C(H)$), necessarily $d_H(c, u) \leq 2$. Consider the 4-point condition on vertices c, u, z, t. We have that $d(c, t) + d(u, z) = rad(G) + 3$ is the largest sum since $d(c, z) + d(u, t) \leq rad(G) + 2$ and $d(c, u) + d(z, t) \leq rad(G)$. However, $d(c, t) + d(u, z) \geq d(c, u) + d(z, t) + 3$, giving a contradiction since the smaller two sums must be equal and differ from the larger one by at most two. Hence, z belongs to $C(H)$ showing that every $c \in C(H)$ has a non-adjacent vertex $z \in C(H)$.

Finally, we show that any such graph H is the center of some distance-hereditary graph G. In what follows, we refer to Fig. 7 for an illustration. If H is a cograph, then one can construct a graph G by simply adding to H four new vertices x, x^*, y, y^*. Vertices x and y are universal to H, and vertices x^*, y^* are pendant to x and y, respectively. Now graph H is the center of G as any vertex u of the cograph H is at most distance 2 to any vertex of G, whereas $d_G(x, y^*) = 3$ and $d_G(y, x^*) = 3$. Suppose now that H is a connected distance-hereditary graph with $diam(H) = 3$ and $C(H)$ is a connected cograph with $rad(C(H)) = 2$. One can construct a graph G by adding to H (with $C(H) = \{c_1, c_2, ..., c_\ell\}$) ℓ new vertices $x_1, x_2, ..., x_\ell$ such that each x_i is pendant to $c_i \in C(H)$. Each $c_i \in C(H)$ has $d_G(c_i, u) \leq 2$ for all $u \in H$. Since $rad(C(H)) = 2$, each c_i has a non-adjacent vertex $c_k \in C(H)$, and therefore $d_G(c_i, x_k) = 3$. Any vertex $u \in H \setminus C(H)$ has a vertex $v \in H \setminus C(H)$, for which $d_G(u, v) = 3$. Furthermore, any such u satisfies $d_G(u, x_i) = d_G(u, c_i) + 1 \leq 3$ for each x_i. Since $rad(C(H)) = 2$, for any pendant x_i, vertex c_i has a non-adjacent vertex $c_k \in C(H)$ and therefore $d_G(x_i, x_k) = 4$. Hence, H is the center of G. □

7. Computing all eccentricities

Recall that Theorem 2 yields a linear time 2-approximation of all eccentricities. We also used ideas from Section 6 to find in linear time a set $S \subseteq C(G)$ sufficient for an additive 1-approximation of all eccentricities. However, it proved to be challenging to get exact computation of all eccentricities using that approach. Instead, we found it more fruitful to use a characteristic pruning sequence of G. We provide in this section a simple linear time algorithm to compute all eccentricities of a distance-hereditary graph G using a weight function and a special pruning sequence produced by processing layers of a breadth-first search tree of G (see [6]).

For a given graph $G = (V(G), E(G))$ and an n-tuple $(p(v_1), p(v_2), ..., p(v_n))$ of non-negative vertex weights, we define the p-weighted eccentricity of each vertex $v \in V(G)$ as $e_{G,p}(v) = \max_{u \in V(G)}\{d_G(v, u) + p(u)\}$. We refer to the set of furthest vertices from a vertex v under weight function p in G as $F_{G,p}(v) = \{u \in V(G) : e_{G,p}(v) = d_G(v, u) + p(u)\}$. Clearly, when $p(v) = 0$ for all $v \in V(G)$, we have $e_{G,p}(v) = e_G(v)$ and $F_{G,p}(v) = F_G(v)$ agreeing with earlier definitions.

Lemma 17. *Let $x, y \in V$ be twins with $p(x) \leq p(y)$. Set $G' = G - \{x\}$ and $p'(v) = p(v)$ for all $v \in V(G')$. Then, $e_{G,p}(v) = e_{G',p'}(v)$ for all $v \in V(G') \setminus \{y\}$, $e_{G,p}(y) = \max\{p(x) + d_G(x, y), e_{G',p'}(y)\}$, and $e_{G,p}(x) = \max\{p(y) + d_G(x, y), e_{G,p}(y)\}$. Moreover, if $F_{G,p}(y) \setminus \{x\} \neq \emptyset$, then $e_{G,p}(y) = e_{G',p'}(y)$.*

Proof. Let $v \in V(G') \setminus \{y\}$. As x and y are twins, $d_G(v, y) = d_G(v, x)$. Since G' is isometric in G, $d_G(v, u) + p(u) = d_{G'}(v, u) + p'(u)$ for any $u \in V(G')$. Then, by definition of eccentricity, $e_{G',p'}(v) = \max_{u \in V(G')}\{d_{G'}(v, u) + p'(u)\} = \max_{u \in V(G) \setminus \{x\}}\{d_G(v, u) + p(u)\}$. As $p(x) \leq p(y)$, $d_G(v, y) + p(y) \geq d_G(v, x) + p(x)$. Thus, $e_{G',p'}(v) = \max\{d_G(v, x) + p(x), \max_{u \in V(G) \setminus \{x\}}\{d_G(v, u) + p(u)\}\} = \max_{u \in V(G)}\{d_G(v, u) + p(u)\} = e_{G,p}(v)$. Similarly, $e_{G,p}(y) = \max\{d_G(x, y) + p(x), \max_{u \in V(G')}\{d_G(y, u) + p(u)\}\} = \max\{d_G(x, y) + p(x), e_{G',p'}(y)\}$. Moreover, if $F_{G,p}(y) \setminus \{x\} \neq \emptyset$, then $e_{G,p}(y) = e_{G',p'}(y)$ as realized by the weighted distance in G' from y to any $v \in F_{G,p}(y) \setminus \{x\}$.

As $d(x, y) \geq 1$ and $p(x) \leq p(y)$, $d_G(x, y) + p(y) \geq \max\{d_G(x, y) + p(x), p(y)\}$. Again, by definition of eccentricity,

$$e_{G,p}(x) = \max\{p(x), d_G(x, y) + p(y), \max_{u \in V(G) \setminus \{x,y\}}\{d_G(x, u) + p(u)\}\}$$

$$= \max\{d_G(x, y) + p(y), \max_{u \in V(G) \setminus \{x,y\}}\{d_G(y, u) + p(u)\}, d_G(x, y) + p(x), p(y)\}$$

$$= \max\{d_G(x, y) + p(y), e_{G,p}(y)\}. \quad \square$$

Lemma 18. *Let x be a vertex pendant to y. Set $G' = G - \{x\}$, $p'(y) = \max\{1 + p(x), p(y)\}$, and $p'(v) = p(v)$ for all $v \in V(G') \setminus \{y\}$. Then, $e_{G,p}(v) = e_{G',p'}(v)$ for all $v \in V(G')$. If $F_{G,p}(y) \setminus \{x\} \neq \emptyset$, then $e_{G,p}(x) = \max\{p(x), e_{G',p'}(y) + 1\}$.*

Proof. Let $v \in V(G')$. As x is pendant to y, $d_G(v, x) = d_G(v, y) + 1$. Since G' is isometric in G, $d_{G'}(v, y) + p'(y) = \max\{d_G(v, y) + 1 + p(x), d_G(v, y) + p(y)\} = \max\{d_G(v, x) + p(x), d_G(v, y) + p(y)\}$. Then, by definition of eccentricity,

$$e_{G',p'}(v) = \max\{d_{G'}(v, y) + p'(y), \max_{u \in V(G') \setminus \{y\}}\{d_{G'}(v, u) + p'(u)\}\}$$

$$= \max\{d_G(v, x) + p(x), d_G(v, y) + p(y), \max_{u \in V(G') \setminus \{y\}}\{d_G(v, u) + p(u)\}\}$$

$$= \max_{u \in V(G)}\{d_G(v, u) + p(u)\} = e_{G,p}(v).$$

Assume now that $F_{G,p}(y) \setminus \{x\} \neq \emptyset$. Hence, $e_{G',p'}(y) = e_{G,p}(y) = \max_{u \in V(G) \setminus \{x\}}\{d_G(y, u) + p(u)\}$. Thus, by definition of eccentricity,

$$e_{G,p}(x) = \max\{p(x), p(y) + 1, \max_{u \in V(G) \setminus \{x,y\}}\{d_G(x, u) + p(u)\}\}$$

$$= \max\{p(x), p(y) + 1, \max_{u \in V(G) \setminus \{x,y\}}\{d_G(y, u) + p(u)\} + 1\}$$

$$= \max\{p(x), \max_{u \in V(G) \setminus \{x\}}\{d_G(y, u) + p(u)\} + 1\} = \max\{p(x), e_{G',p'}(y) + 1\}. \quad \square$$

We use the pruning sequence (the vertex elimination ordering) $\sigma = (v_1, ..., v_n)$ that can be constructed in linear time via ρ iterations of a systematic removal of pendants/twins from each layer $\mathcal{L}_\rho, ..., \mathcal{L}_1$ of a breadth-first search tree rooted at v_n (see [6]). Iteration k, where $k = \rho, ..., 1$, consists of four consecutive steps:
  (a) remove any $x \in \mathcal{L}_k$ twin to some $y \in \mathcal{L}_k$ of the same connected component in $\mathcal{L}_k$ (i.e., x and y belong to the same connected component of the subgraph of G induced by vertices of $\mathcal{L}_k$),
  (b) remove any $x \in \mathcal{L}_k$ pendant to some $y \in \mathcal{L}_{k-1}$,
  (c) remove any $x \in \mathcal{L}_{k-1}$ twin to some $y \in \mathcal{L}_{k-1}$ belonging to the same neighborhood $N(z) \cap \mathcal{L}_{k-1}$ of some $z \in \mathcal{L}_k$, and
  (d) remove any $x \in \mathcal{L}_k$ pendant to some $y \in \mathcal{L}_{k-1}$.
Note that we move to the next step only when no vertex remains satisfying the condition of the previous step. At the end of iteration k, all vertices of $\mathcal{L}_k$ have been removed (see [6]). By this ordering, any $u \in \mathcal{L}_k$ satisfies that if u is a pendant to v, then $v \in \mathcal{L}_{k-1}$, and if u is a twin to v, then $v \in \mathcal{L}_k$. Let G_i denote the graph induced by $\{v_i, ..., v_n\}$ for each $i = 1, ..., n$.

Theorem 4. *There is a linear time algorithm to compute all eccentricities in a distance-hereditary graph.*

Proof. Let $\sigma = (v_1, ..., v_n)$ be the pruning sequence constructed as described above by each iteration $k = rad(G), ..., 1$ of removing vertices from layer $\mathcal{L}_k$ of a BFS tree rooted at a central vertex v_n. Denote by $v_y \in \mathcal{L}_2$ the first pendant vertex of σ encountered in step (d) of iteration 2 (or in step (b) if $\mathcal{L}_2$ becomes empty after steps (a) and (b)). Denote by v_z the first vertex of σ encountered in iteration 1. Thus, the graph G_z consists of v_n and some twins/pendants in $\mathcal{L}_1$ adjacent to v_n. The algorithm is summarized as follows. We process vertices v_i, $i < z$, from v_1 to v_z (from left to right along σ). We denote

by p_i the weight function of each vertex immediately before vertex v_i is processed. For each vertex $v_j \in \sigma$, set $p_1(v_j) = 0$. As each v_i is processed, p_{i+1} is invariant (that is, $p_{i+1}(v_j) = p_i(v_j)$ for every v_j) with the exception of one case: if v_i is a pendant to v_j in G_i, then let $p_{i+1}(v_j) = \max\{p_i(v_i) + 1, p_i(v_j)\}$. We can assume that if v_i is a twin to vertex v_j in G_i, then $p_i(v_i) \leq p_i(v_j)$, since otherwise, their positions as twins can be swapped in σ. Observe that the weight function of a vertex $v_j \in \mathcal{L}_k$ can only increase when a vertex $v_i \in \mathcal{L}_{k+1}$ is pendant to v_j, where $i < j$. Hence, if any vertex v_i belongs to layer $\mathcal{L}_k$, then for any integer ℓ, $p_\ell(v_i) \leq rad(G) - k$. Additionally, every vertex v_i and integers $\ell < \kappa$ satisfy $p_\ell(v_i) \leq p_\kappa(v_i)$. After all vertices v_i, $i < z$, are processed (along σ from left to right), we next compute all p_z-weighted eccentricities in G_z. Then, we compute all p_y-weighted eccentricities in G_y. Finally, we process each vertex v_i, $i < y$, along the reverse direction of σ. The p_{i+1}-weighted eccentricities in G_{i+1} are used to obtain the p_i-weighted eccentricities in G_i.

Backward phase 1: Compute all p_z-weighted eccentricities in G_z. Denote by V^* and N_i^* the vertex lists ordered by decreasing weight p_z from the respective vertex sets $V(G_z)$ and $N(v_i) \cap V(G_z)$ for each $v_i \in V(G_z)$. The lists are ordered in total linear time with a bucket sort. The first vertex w of N_i^* has maximum $p_z(w)$ among neighbors of v_i in G_z, and the first vertex u of $V^* \setminus N_i^*$ has maximum $p_z(u)$ among non-neighbors of v_i. By definition, the weighted eccentricity of each v_i is $e_{G_z, p_z}(v_i) = \max\{p_z(v_i), 1 + p_z(w), 2 + p_z(u)\}$ if there exists a non-neighbor $u \in V^* \setminus N_i^*$, and $e_{G_z, p_z}(v_i) = \max\{p_z(v_i), 1 + p_z(w)\}$ otherwise.

Backward phase 2: Compute all p_y-weighted eccentricities in G_y. By choice of y, each vertex v_i for $y \leq i < z$ satisfies $v_i \in \mathcal{L}_2$ and v_i is pendant to a vertex $v_j \in \mathcal{L}_1$ in G_y. Hence, for $M = \{v_z, ..., v_n\}$ and $S = \{v_y, ..., v_{z-1}\}$, $\max_{m \in M} p_z(m) = \max\{\max_{m \in M} p_y(m), \max_{u \in S} p_y(u) + 1\}$ holds. Therefore, we may again use ideas from the previous phase. For each vertex v_i, $y \leq i < z$, pendant to v_j, we have a vertex $w \in N(v_j)$ in G_z with maximum $p_z(w)$ among neighbors of v_j in G_z and a vertex $u \notin N(v_j)$ in G_z with maximum $p_z(u)$ among non-neighbors of v_j in G_z. By definition, the weighted eccentricity of each v_i is $e_{G_y, p_y}(v_i) = \max\{p_i(v_i), p_i(v_j) + 1, p_z(w) + 2, p_z(u) + 3\}$ if there exists in G_z a non-neighbor $u \notin N(v_j)$, and $e_{G_y, p_y}(v_i) = \max\{p_i(v_i), p_z(v_j) + 1, p_z(w) + 2\}$ otherwise. Additionally, by Lemma 18, each vertex v_ℓ for $\ell \geq z$ has equal p_z and p_y weighted eccentricities, that is, $e_{G_y, p_y}(v_\ell) = e_{G_z, p_z}(v_\ell)$.

Backward phase 3: Compute all p_i-weighted eccentricities in G_i for $i < y$ along a reverse direction of σ. If v_i is a twin to v_j in G_i, by Lemma 17, $e_{G_i, p_i}(v_j) = \max\{p_i(v_i) + d_{G_i}(v_i, v_j), e_{G_{i+1}, p_{i+1}}(v_j)\}$, $e_{G_i, p_i}(v_i) = \max\{p_i(v_j) + d_{G_i}(v_i, v_j), e_{G_i, p_i}(v_j)\}$, and for all $u \in V(G_i) \setminus \{v_i, v_j\}$ $e_{G_i, p_i}(u) = e_{G_{i+1}, p_{i+1}}(u)$. We consider now the case that v_i is a pendant to v_j. By Lemma 18, each $u \in V(G_i) \setminus \{v_i\}$ satisfy $e_{G_i, p_i}(u) = e_{G_{i+1}, p_{i+1}}(u)$. We claim that $e_{G_i, p_i}(v_i) = \max\{p_i(v_i), e_{G_{i+1}, p_{i+1}}(v_j) + 1\}$. It remains only to show that any pendant v_i, where $i < y$, satisfies $F_{G_i, p_i}(v_j) \setminus \{v_i\} \neq \emptyset$; applying Lemma 18 then proves the claim.

By contradiction, let $v_i \in \mathcal{L}_\gamma$ pendant to $v_j \in \mathcal{L}_{\gamma-1}$ be the earliest vertex in σ with $i < y$ and $F_{G_i, p_i}(v_j) = \{v_i\}$. Hence, $e_{G_i, p_i}(v_j) = d_{G_i}(v_j, v_i) + p_i(v_i) > \max_{i+1 \leq t \leq n}\{d_{G_i}(v_j, v_t) + p_i(v_t)\}$. As $i < y$ and $v_y \in \mathcal{L}_2$, then $\gamma \geq 2$. Thus, $p_i(v_i) \leq rad(G) - 2$ and $e_{G_i, p_i}(v_j) = d_{G_i}(v_j, v_i) + p_i(v_i) \leq rad(G) - 1$. If $e_{G_i, p_i}(v_j) = e_G(v_j)$, we obtain a contradiction with $e_G(v_j) \leq rad(G) - 1$. Therefore, $e_{G_i, p_i}(v_j) < e_G(v_j)$. Let $v_\ell \in \sigma$ be the earliest vertex such that $e_{G_\ell, p_\ell}(v_j) > e_{G_{\ell+1}, p_{\ell+1}}(v_j)$, where $\ell < i < j$. By Lemma 17 and Lemma 18, v_ℓ is a twin to v_j in G_ℓ such that $F_{G_\ell, p_\ell}(v_j) = \{v_\ell\}$ and $p_\ell(v_\ell) \leq p_\ell(v_j)$. Then, $rad(G) \leq e_G(v_j) = e_{G_\ell, p_\ell}(v_j) = d_G(v_\ell, v_j) + p_\ell(v_\ell)$. Hence, $p_i(v_j) \geq p_\ell(v_j) \geq p_\ell(v_\ell) \geq rad(G) - d_G(v_\ell, v_j)$. If $p_i(v_j) = rad(G) - 1$ or if v_ℓ is a true twin to v_j, then $p_i(v_j) \geq rad(G) - 1$, and a contradiction arises with $p_i(v_j) < d_{G_i}(v_j, v_i) + p_i(v_i) \leq rad(G) - 1$ (recall that $F_{G_i, p_i}(v_j) = \{v_i\}$). Necessarily, v_ℓ is a false twin to v_j and $p_i(v_j) = rad(G) - 2$. As $rad(G) - 2 = p_i(v_j) < e_{G_i, p_i}(v_j) = d_G(v_i, v_j) + p_i(v_i) \leq 1 + rad(G) - \gamma$, we obtain $\gamma < 3$. Hence, $v_i \in \mathcal{L}_2$ and $v_\ell \in \mathcal{L}_1$. As $\ell < i$, necessarily v_ℓ is removed in iteration 2 step (c) of σ construction. However, this implies that v_i is removed in iteration 2 step (d); therefore, $i \geq y$, a contradiction that proves the claim. □

8. Concluding remarks

We have shown that the eccentricity function in distance-hereditary graphs is almost unimodal. We used this result to fully characterize centers of distance-hereditary graphs and to provide several bounds on the eccentricity of a vertex. Finally, a new linear time algorithm to calculate all eccentricities is presented.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

References

[1] Hend Alrasheed, Feodor F. Dragan, Core–periphery models for graphs based on their δ-hyperbolicity: An example using biological networks, J. Algorithms Comput. Technol. 11 (1) (Sep 2016) 40–57.
[2] Hans-Jürgen Bandelt, Henry Martyn Mulder, Distance-hereditary graphs, J. Comb. Theory, Ser. B 41 (2) (October 1986) 182–208.

[3] V.D. Chepoi, Centers of triangulated graphs, Math. Notes Acad. Sci. USSR 43 (1) (Jan 1988) 82–86.
[4] Victor Chepoi, Feodor F. Dragan, Michel Habib, Yann Vaxès, Hend Alrasheed, Fast approximation of eccentricities and distances in hyperbolic graphs, J. Graph Algorithms Appl. 23 (2) (2019) 393–433.
[5] David Coudert, Guillaume Ducoffe, Alexandru Popa, Fully polynomial FPT algorithms for some classes of bounded clique-width graphs, ACM Trans. Algorithms 15 (3) (Jun 2019) 1–57.
[6] Guillaume Damiand, Michel Habib, Christophe Paul, A simple paradigm for graph recognition: application to cographs and distance hereditary graphs, Theor. Comput. Sci. 263 (1) (2001) 99–111.
[7] A. D'Atri, M. Moscarini, Distance-hereditary graphs, steiner trees, and connected domination, SIAM J. Comput. 17 (3) (1988) 521–538.
[8] Feodor F. Dragan, Centers of Graphs and the Helly Property, (in Russian), PhD thesis, Moldava State University, Chişinău, 1989.
[9] Feodor F. Dragan, Dominating cliques in distance-hereditary graphs, Lect. Notes Comput. Sci. (1994) 370–381.
[10] Feodor F. Dragan, An eccentricity 2-approximating spanning tree of a chordal graph is computable in linear time, Inf. Process. Lett. 154 (2020).
[11] Feodor F. Dragan, Heather M. Guarnera, Eccentricity terrain of δ-hyperbolic graphs, J. Comput. Syst. Sci. (2020).
[12] Feodor F. Dragan, Michel Habib, Laurent Viennot, Revisiting radius, diameter, and all eccentricity computation in graphs through certificates, CoRR, abs/1803.04660, 2018.
[13] Feodor F. Dragan, Ekkehard Köhler, Hend Alrasheed, Eccentricity approximating trees, Discrete Appl. Math. 232 (2017) 142–156.
[14] Feodor F. Dragan, Falk Nicolai, LexBFS-orderings of distance-hereditary graphs with application to the diametral pair problem, Discrete Appl. Math. 98 (3) (2000) 191–207.
[15] Feodor F. Dragan, Falk Nicolai, Andreas Brandstädt, Convexity and HHD-Free graphs, SIAM J. Discrete Math. 12 (1999) 119–135.
[16] Edward Howorka, A characterization of distance-hereditary graphs, Q. J. Math. 28 (4) (1977) 417.
[17] Jordan Camille, Sur les assemblages de lignes, J. Eeine Angew. Math. (Crelles J.) 1869 (70) (Jan 1869) 185–190.
[18] Dirk Koschützki, Katharina Anna Lehmann, Leon Peeters, Stefan Richter, Dagmar Tenfelde-Podehl, Oliver Zlotowski, Centrality indices, in: Network Analysis, Springer, 2005, pp. 16–61.
[19] Erich Prisner, Distance approximating spanning trees, in: STACS 97, 1997, pp. 499–510.
[20] Erich Prisner, Eccentricity-approximating trees in chordal graphs, Discrete Math. 220 (2000) 263–269.
[21] Andrzej Proskurowski, Centers of 2–trees, Ann. Discrete Math. 9 (1980) 1–5.
[22] Andrzej Proskurowski, Centers of maximal outerplanar graphs, J. Graph Theory 4 (1) (1980) 75–79.
[23] Hong-Gwa Yeh, Gerard J. Chang, Centers and medians of distance-hereditary graphs, Discrete Math. 265 (1) (2003) 297–310.
[24] S.V. Yushmanov, V. Chepoi, A general method of investigation of metric graph properties related to the eccentricity, (in Russian), Math. Probl. Cybern. 3 (1991) 217–232.