Collective Tree Spanners of Unit Disk Graphs with Applications to Compact and Low Delay Routing Labeling Schemes

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(joint work with Yang Xiang and Chenyu Yan)

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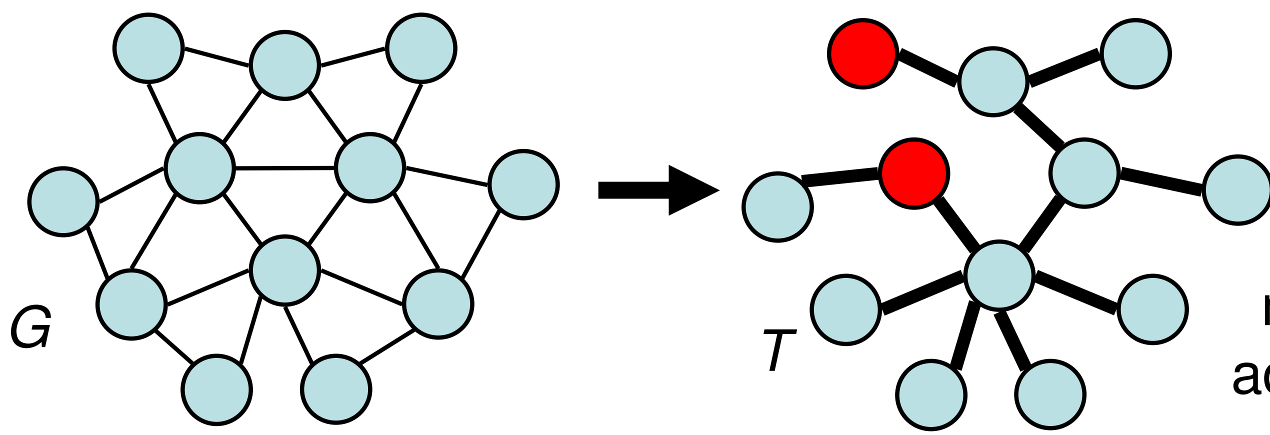
Well-known Tree *t*-Spanner Problem

Given unweighted undirected graph $G=(V,E)$ and integers t,r.
Does G admit a spanning tree $T=(V,E')$ such that

$\forall u,v \in V,\ \ dist_T(v,u) \le t \times dist_G(v,u)$ (a *multiplicative* tree *t-spanner* of G)

or

$\forall u,v \in V,\ \ dist_T(u,v) - dist_G(u,v) \le r$ (an *additive* tree *r-spanner* of G)?



multiplicative tree 4-,
additive tree 3-spanner
of G

Some known results for the tree spanner problem

(mostly multiplicative case)

- general graphs [CC'95]
  - $t \geq 4$ is NP-complete. (t=3 is still open, $t \leq 2$ is P)
- approximation algorithm for general graphs [EP'04]
  - $O(\log n)$ approximation algorithm
- chordal graphs [BDLL'02]
  - $t \geq 4$ is NP-complete. (t=3 is still open.)
- planar graphs
  - arbitrary $t \geq 4$, is NP-complete. (t=3 is polyl time solvable.) [FK'01]
  - for each fixed t, is linear time solvable [DFG'08]
- easy to construct for some special families of graphs.

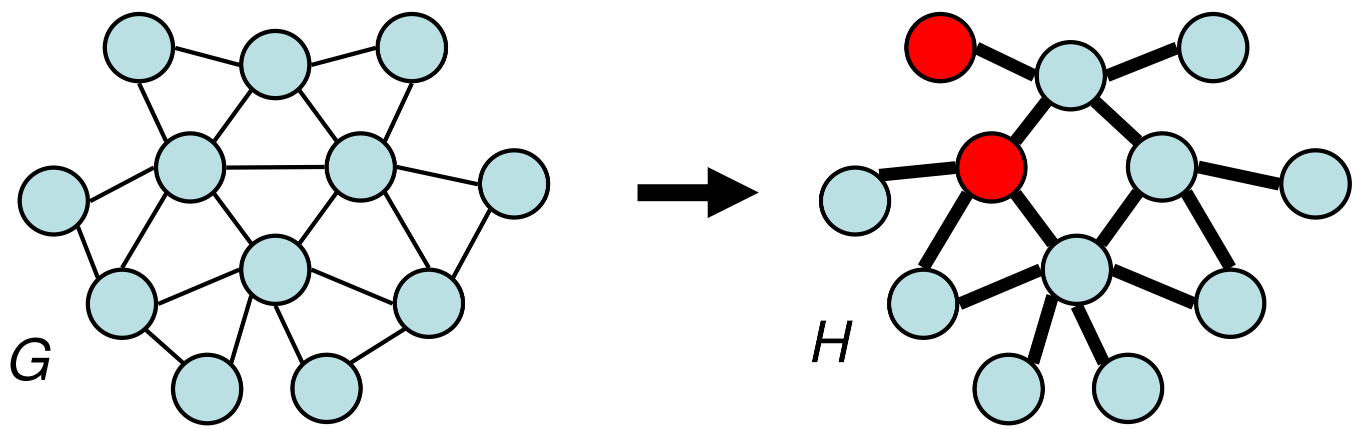
Well-known *Sparse t-Spanner Problem*

Given unweighted undirected graph *G=(V,E)* and integers *t,m,r.*
Does *G* admit a spanning graph *H =(V,E')* with *|E'|* $\leq$ *m* s.t.

$$\forall u,v \in V, \quad dist_H(v,u) \leq t \times dist_G(v,u)$$ (a *multiplicative t-spanner* of *G*)

or

$$\forall u,v \in V, \quad dist_H(u,v) - dist_G(u,v) \leq r$$ (an *additive r-spanner* of *G*)?



multiplicative 2- and additive 1-spanner of *G*

Some known results for sparse spanner problems

- general graphs
  - *t, m*≥*1* is NP-complete [PS'89]
  - multiplicative (2k-1)-spanner with $n^{1+1/k}$ edges [TZ'01, BS'03]
- *n*-vertex chordal graphs (multiplicative case) [PS'89]
  (*G* is chordal if it has no chordless cycles of length >3)
  - multiplicative 3-spanner with *O(n* log*n)* edges
  - multiplicative 5-spanner with *2n-2* edges
- *n*-vertex *c*-chordal graphs (additive case) [CDY'03, DYL'04]
  (*G* is *c*-chordal if it has no chordless cycles of length >c*)*
  - additive (c+1)-spanner with *2n-2* edges
  - additive $(2\lfloor c/2 \rfloor)$-spanner with *n log n* edges
  - For chordal graphs: additive *4*-spanner with *2n-2* edges, additive *2*-spanner with *n log n* edges
- planar graphs
  - If *t, m*≥*1* are constants, then in linear time [DFG'08]
  - PTAS for every *t*≥*1* [DFG'08]

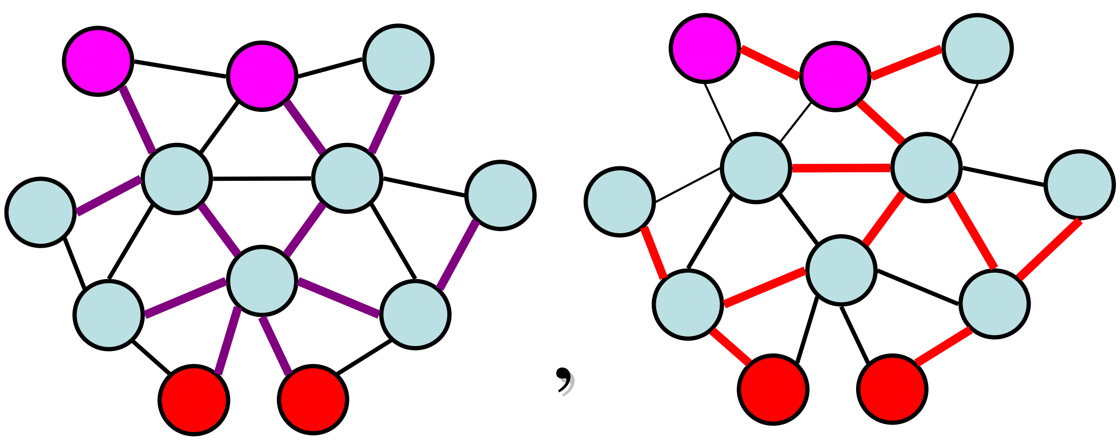
Relatively new Collective Additive Tree r -Spanners Problem

Given unweighted undirected graph *G=(V,E)* and integers μ, *r.*
Does *G* admit a system of μ collective additive tree *r-spanners* $\{T_1, T_2, \ldots, T_\mu\}$ such that

$$\forall u, v \in V \; and \; \exists 0 \le i \le \mu, \; dist_{T_i}(v,u) - dist_G(v,u) \le r$$

(a system of μ collective *additive tree r-spanners* of *G*)?

surplus



2 **collective additive tree *2*-spanners**

collective multiplicative
tree t-spanners
can be defined similarly

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Relatively new Collective Additive Tree r-Spanners Problem

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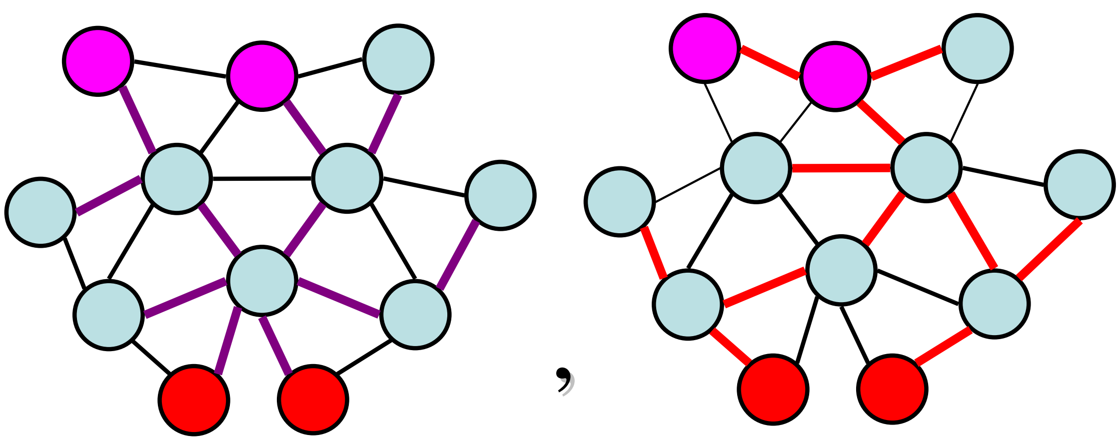
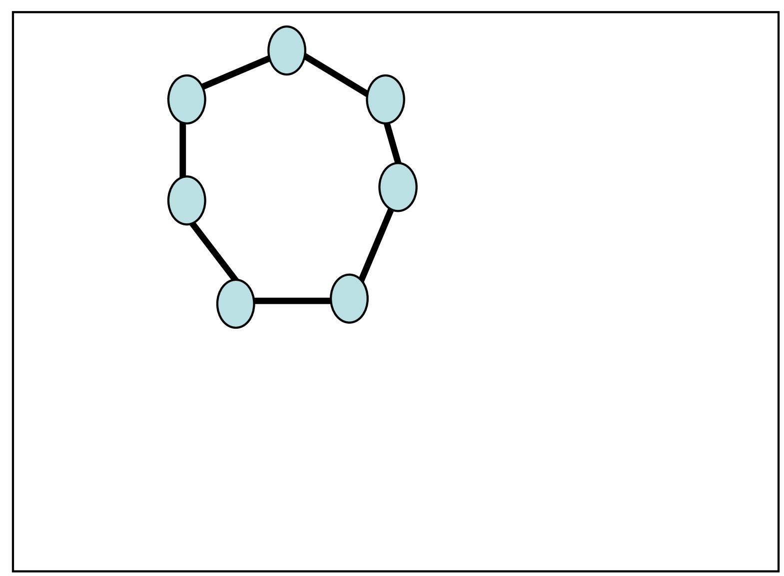


2 collective additive tree 2-spanners



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Relatively new *Collective Additive Tree r-Spanners Problem*

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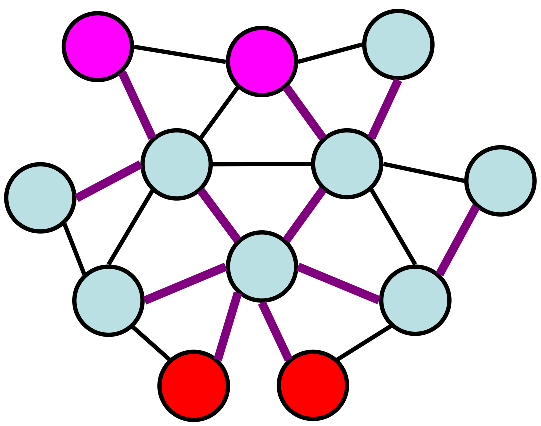
(a system of μ collective *additive tree r-spanners* of *G*)?



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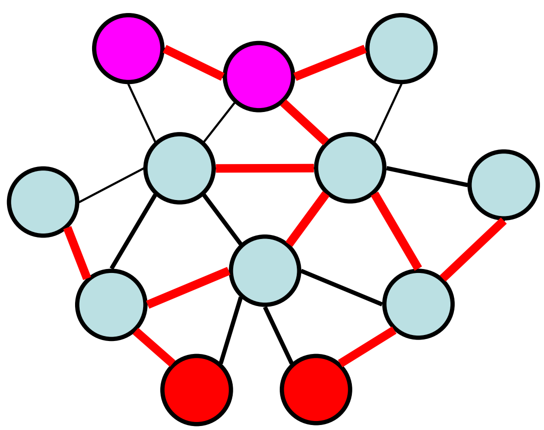
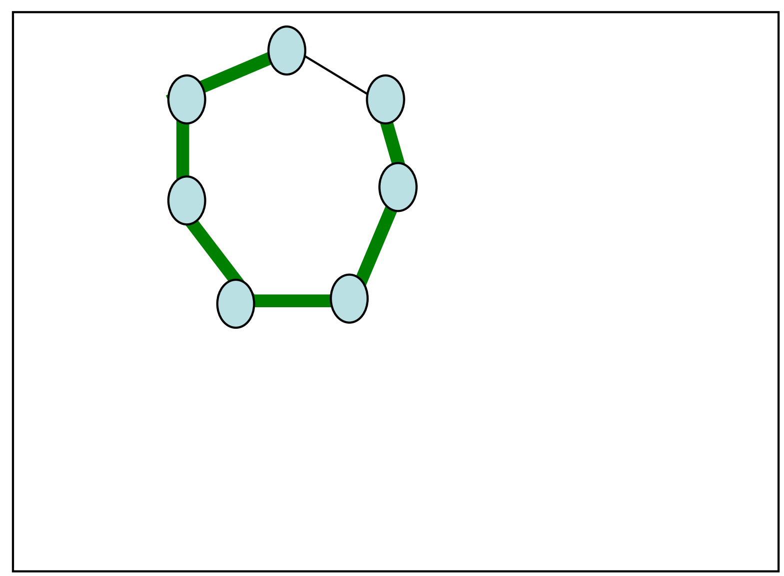


***2* collective additive tree *2*-spanners**



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Relatively new *Collective Additive Tree r-Spanners Problem*

Given unweighted undirected graph *G=(V,E)* and integers μ, *r.*
Does *G* admit a system of μ collective additive tree *r-spanners* $\{T_1, T_2 \ldots, T_\mu\}$ such that

$$\forall\, u, v \in V \;and\; \exists 0 \le i \le \mu,\; dist_{T_i}(v,u) - dist_G(v,u) \le r$$

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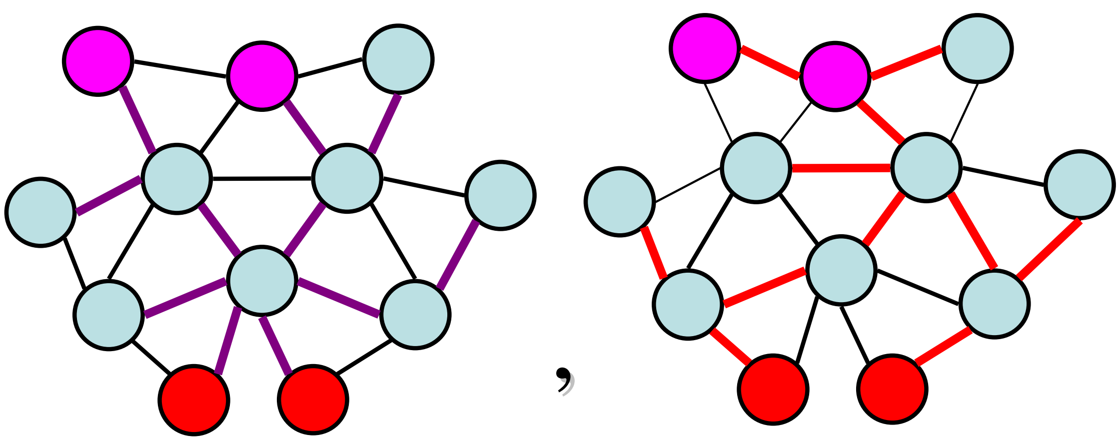
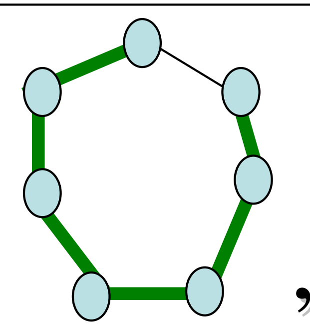
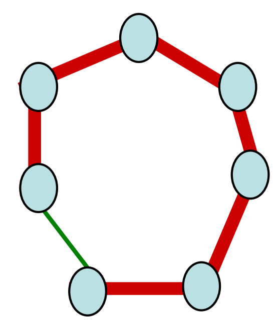


2 **collective additive tree *2*-spanners**





2 **collective additive tree *0*-spanners or multiplicative tree 1-spanners**

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Applications of Collective Tree Spanners

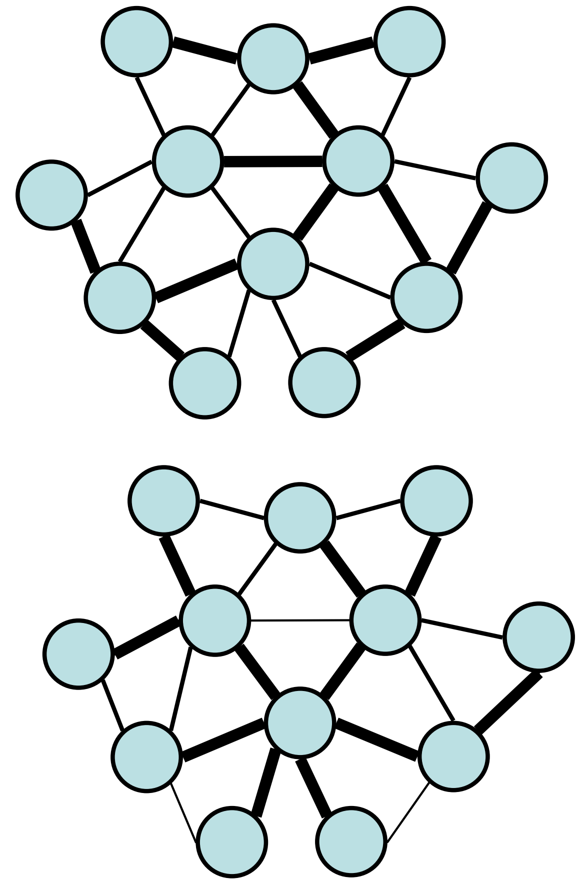
- message routing in networks

  Efficient routing schemes are known for trees but not for general graphs. For any two nodes, we can route the message between them in one of the trees which approximates the distance between them.

  - ($\mu \log^2 n$)-bit labels,
  - $O(\mu)$ initiation, $O(1)$ decision

- solution for sparse *t*-spanner problem

  If a graph admits a system of μ collective additive tree *r*-spanners, then the graph admits a sparse additive *r*-spanner with at most $\mu(n-1)$ edges, where *n* is the number of nodes.



2 collective tree 2-spanners for *G*

Previous results on the collective tree spanners problem

(Dragan, Yan, Lomonosov [SWAT'04])

(Corneil, Dragan, Köhler, Yan [WG'05])

- chordal graphs, chordal bipartite graphs
  - log *n* collective additive tree *2*-spanners in polynomial time
  - $\Omega(n^{1/2})$ or $\Omega(n)$ trees necessary to get +1
  - no constant number of trees guaranties +2 (+3)
- circular-arc graphs
  - 2 collective additive tree *2*-spanners in polynomial time
- *c*-chordal graphs
  - log *n* collective additive tree $2\lfloor c/2 \rfloor$-spanners in polynomial time
- interval graphs
  - log *n* collective additive tree 1-spanners in polynomial time
  - no constant number of trees guaranties +1

Previous results on the collective tree spanners problem

(Dragan, Yan, Corneil [WG'04])

- AT-free graphs
  - include: interval, permutation, trapezoid, co-comparability
  - 2 collective additive tree *2*-spanners in linear time
  - an additive tree *3*-spanner in linear time (before)
- graphs with a dominating shortest path
  - an additive tree *4*-spanner in polynomial time (before)
  - 2 collective additive tree *3*-spanners in polynomial time
  - 5 collective additive tree *2*-spanners in polynomial time
- graphs with asteroidal number an(G)=k
  - k(k-1)/2 collective additive tree 4-spanners in polynomial time
  - k(k-1) collective additive tree 3-spanners in polynomial time

Previous results on the collective tree spanners problem

(Gupta, Kumar,Rastogi [SICOMP'05])

- the only paper (before) on collective multiplicative tree spanners in weighted planar graphs
- any weighted planar graph admits a system of *O(log n)* collective multiplicative tree 3-spanners
- they are called there the tree-covers.
- it follows from (Corneil, Dragan, Köhler, Yan [WG'05]) that
  - no constant number of trees guaranties +c (for any constant c)

Some results on collective additive tree spanners of weighted graphs with bounded parameters

(Dragan, Yan [ISAAC'04])

Graph class	μ	r
planar	$O(\sqrt{n})$	*0*
with genus g	$O(\sqrt{gn})$	*0*
W/o an h-vertex minor	$O(\sqrt{h^3 n})$	*0*
$tw(G) \leq k\text{-}1$	$k \log_2 n$	*0*
$cw(G) \leq k$	$k \log_{3/2} n$	2w
c-chordal	*next*	*slide*

$\Omega(\sqrt{n} \log\log n / \log^2 n)$ to get +0 (planar, with genus g)

No constant number of trees guaranties $+r$ for any constant r even for outer-planar graphs ($tw(G) \leq k\text{-}1$)

$\Omega(n)$ to get +1 ($tw(G) \leq k\text{-}1$, $cw(G) \leq k$)

- w is the length of a longest edge in G

Some results on collective additive tree spanners of weighted c-chordal graphs

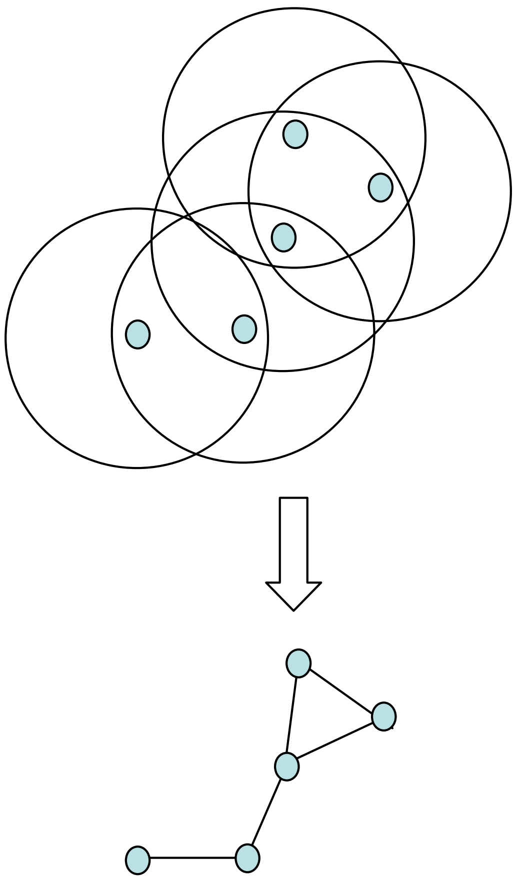
(Dragan, Yan [ISAAC'04])

Graph class	μ	r
c-chordal ***(c>4)***	$\log_2 n$ $4\log_2 n$ $5\log_2 n$	$2\left\lfloor \frac{c}{2} \right\rfloor w$ $2(\left\lfloor \frac{c}{3} \right\rfloor + 1)w$ $2\left\lfloor \frac{c+2}{3} \right\rfloor w$
4-chordal	$6\log_2 n$	$2w$
weakly chordal	$4\log_2 n$	$2w$

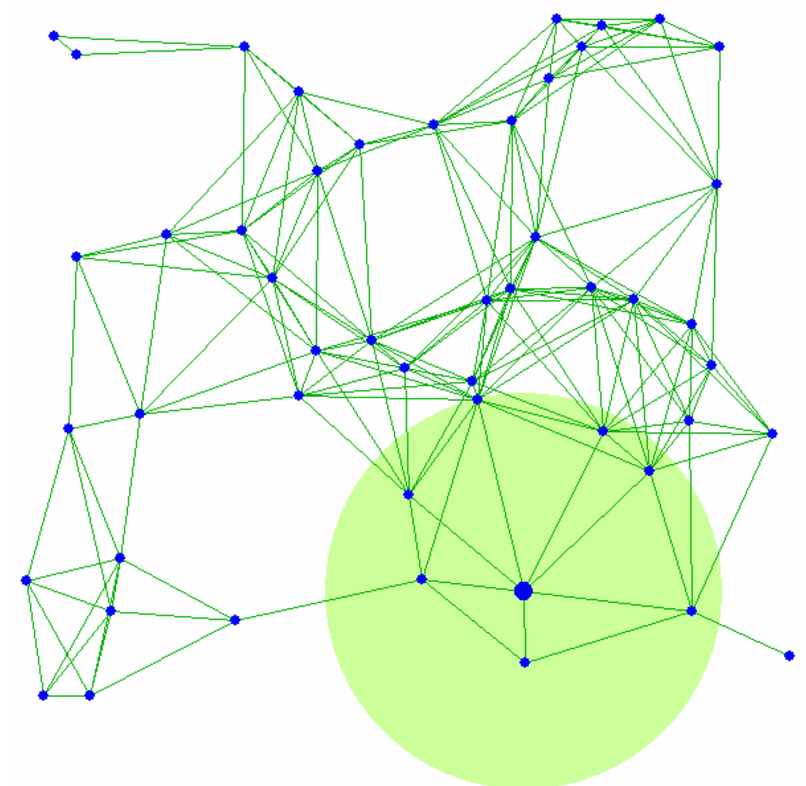
No constant number of trees guaranties $+r$ for any constant r even for weakly chordal graphs

Unit Disk Graphs

- Unit Disk Graphs are the intersection graphs of equal sized circles in the plane.

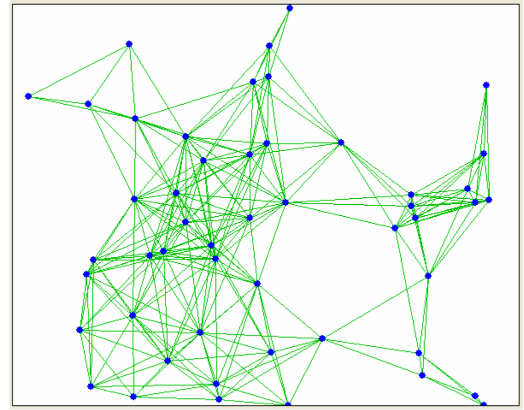


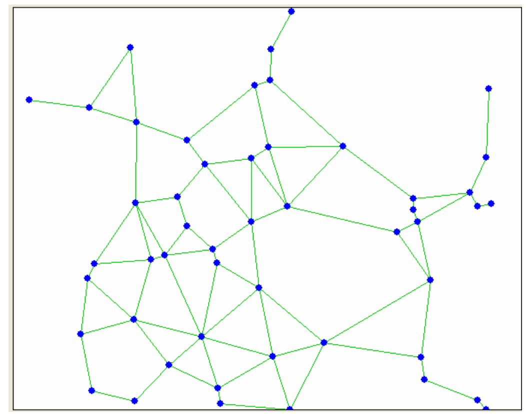
Model
wireless
networks

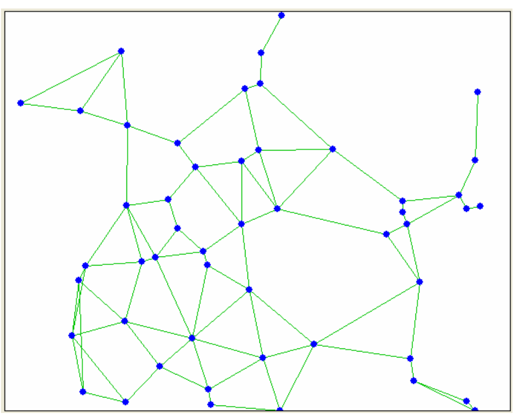


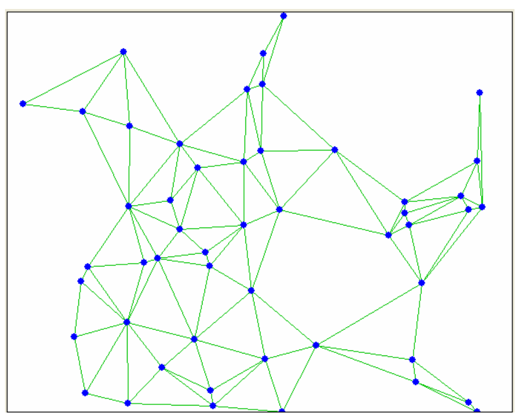
Known Sparse spanners for UDGs

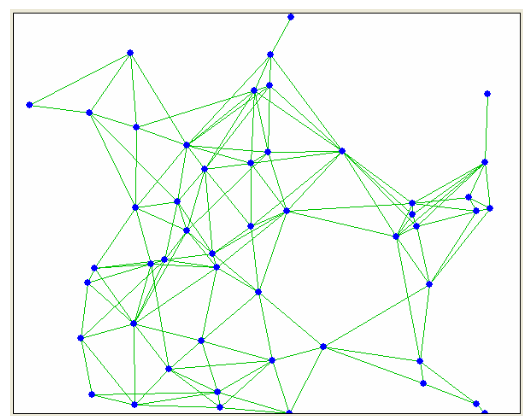
Spanner	Length Stretch Factor	Hop Stretch Factor
Gabriel Graph	$\frac{4\pi\sqrt{2n-4}}{3}$	-
Yao Graph	$\frac{1}{1-2\sin\frac{\pi}{k}}$	-
Unit Del	$\frac{4\sqrt{3}}{9}\pi$	-
LDel	$\frac{4\sqrt{3}}{9}\pi$	**Very Large Constant**











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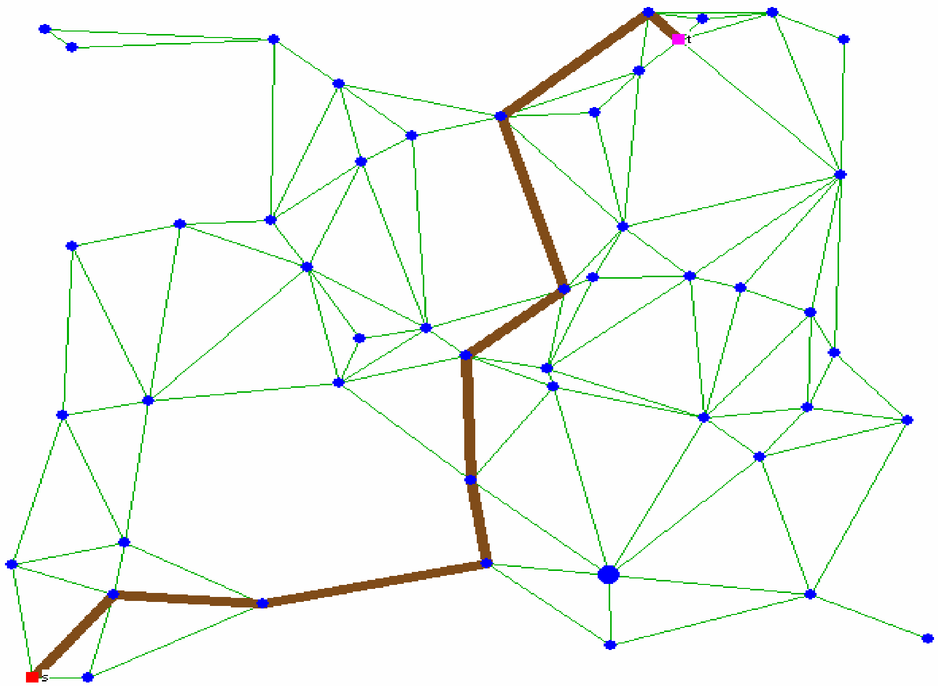
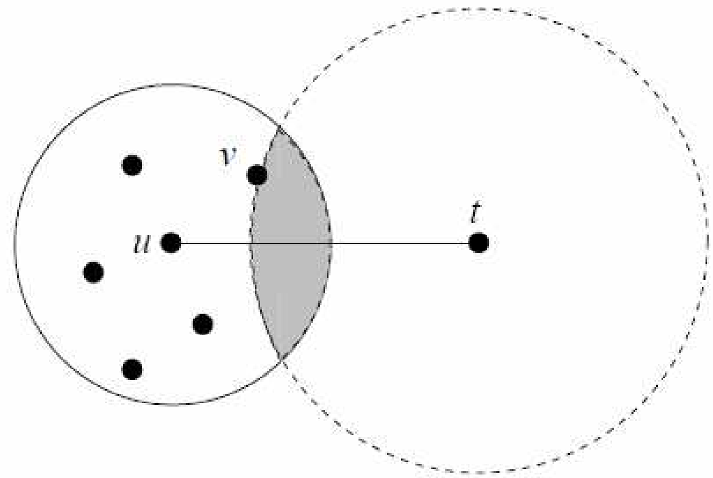
Unit Delaunay Triangulation and Greedy Routing





- [KG'92] showed that Unit Delaunay triangulation is a length t-spanner for t≈2.42.
- (Localized) Unit Delaunay triangulation with Greedy Routing (no guarantee of delivery).
- Face greedy routing by [BMSU'99] guarantees delivery (*4m* moves)

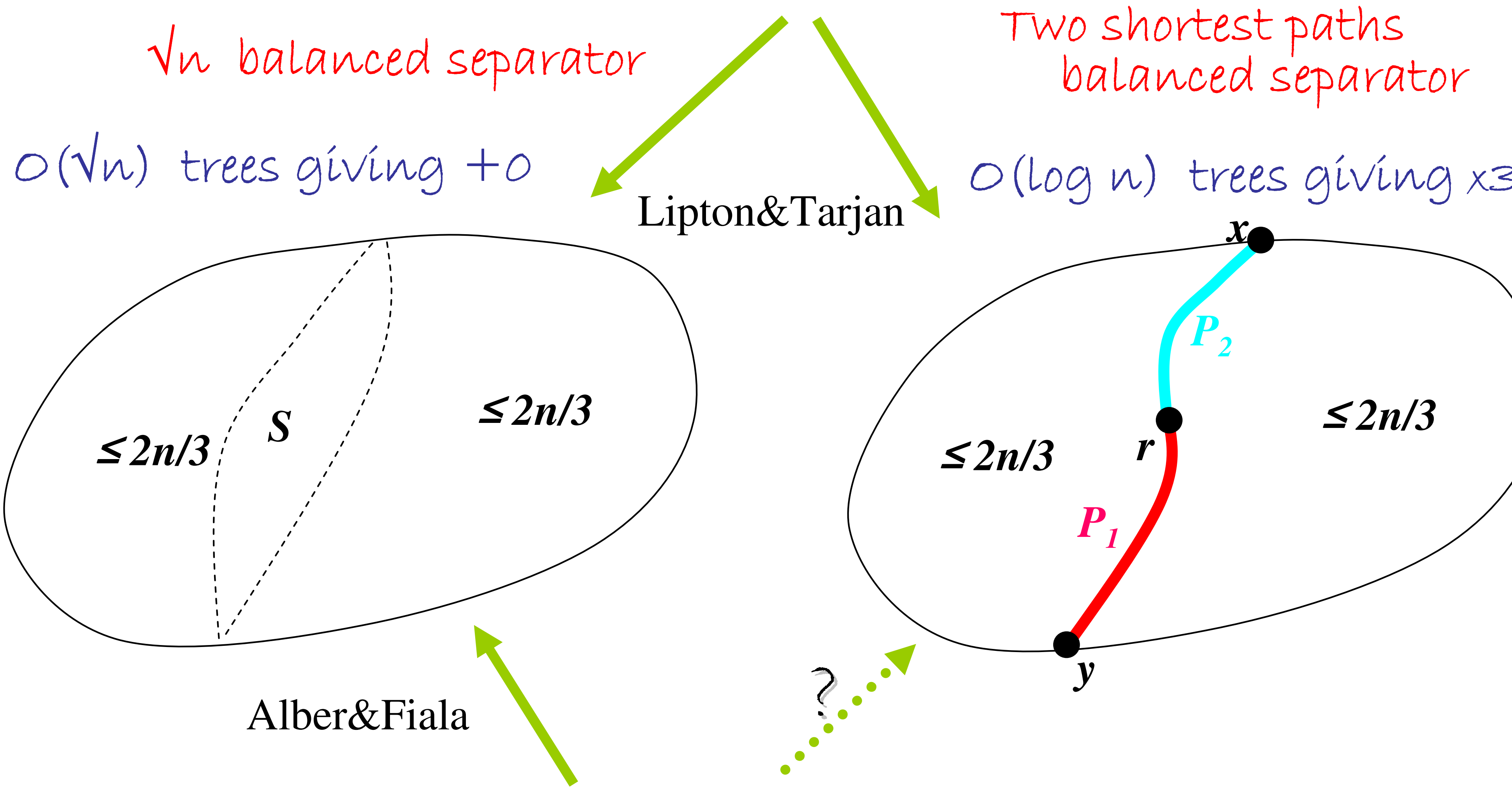
New results on collective tree spanners of Unit Disk Graphs

Definition: A graph G admits a system of μ collective tree (t, r)-spanners if there is a system $\mathcal{T}(G)$ of at most μ spanning trees of G such that for any two vertices x, y of G a spanning tree $T \in \mathcal{T}(G)$ exists such that

$$d_T(x,y) \leq t\, d_G(x,y)+r.$$

Theorem: Any Unit Disk Graph admits a system of at most $2\log_{3/2}n+2$ collective tree $(3,12)$-spanners. Construction is in $O((C+m)\log n)$ time where C is the number of crossings in G.

Planar Graphs



Unit Disk Graphs

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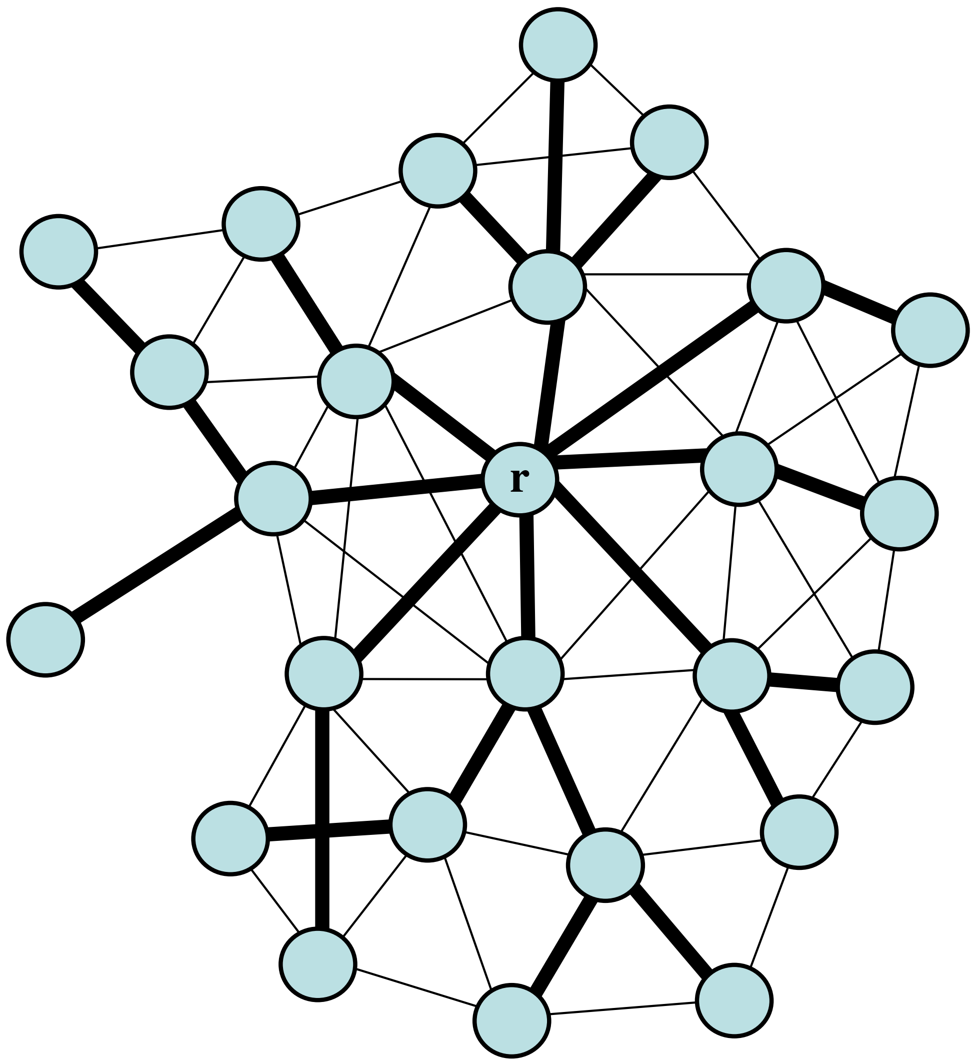
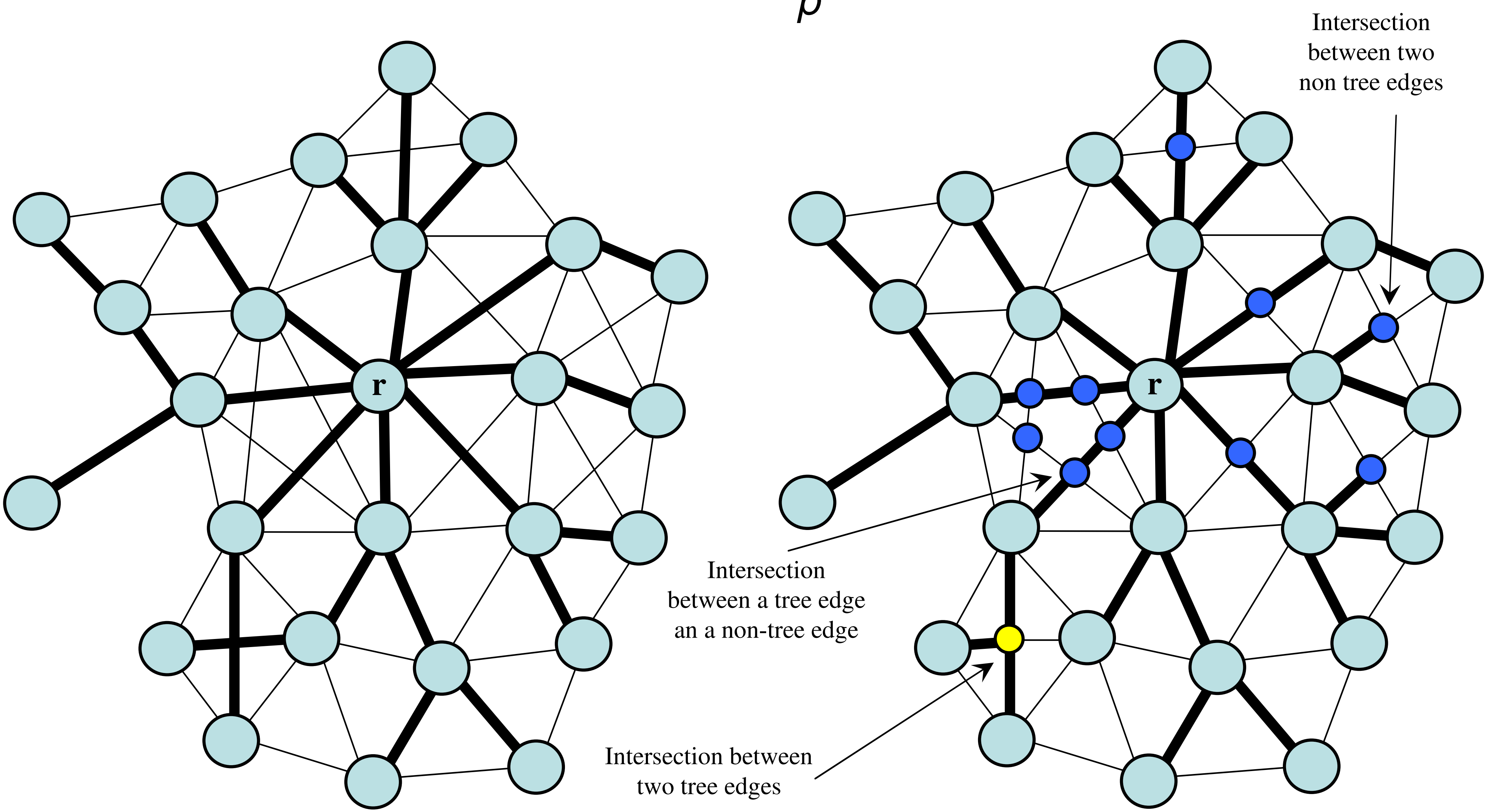
Finding a Balanced Separator in a Unit Disk Graph

1. Build a layering spanning tree T for G.
2. Convert the Unit Disk Graph G into a planar graph G_p and T into a spanning tree T_p for G_p.
3. Apply Lipton&Tarjan's separator theorem to the planar graph G_p and spanning tree T_p to find a balanced separator S_p for G_p.
4. (The most important Step) From S_p, reconstruct a balanced separator S for G.

Step 1: Build a layering spanning tree *T* for *G*

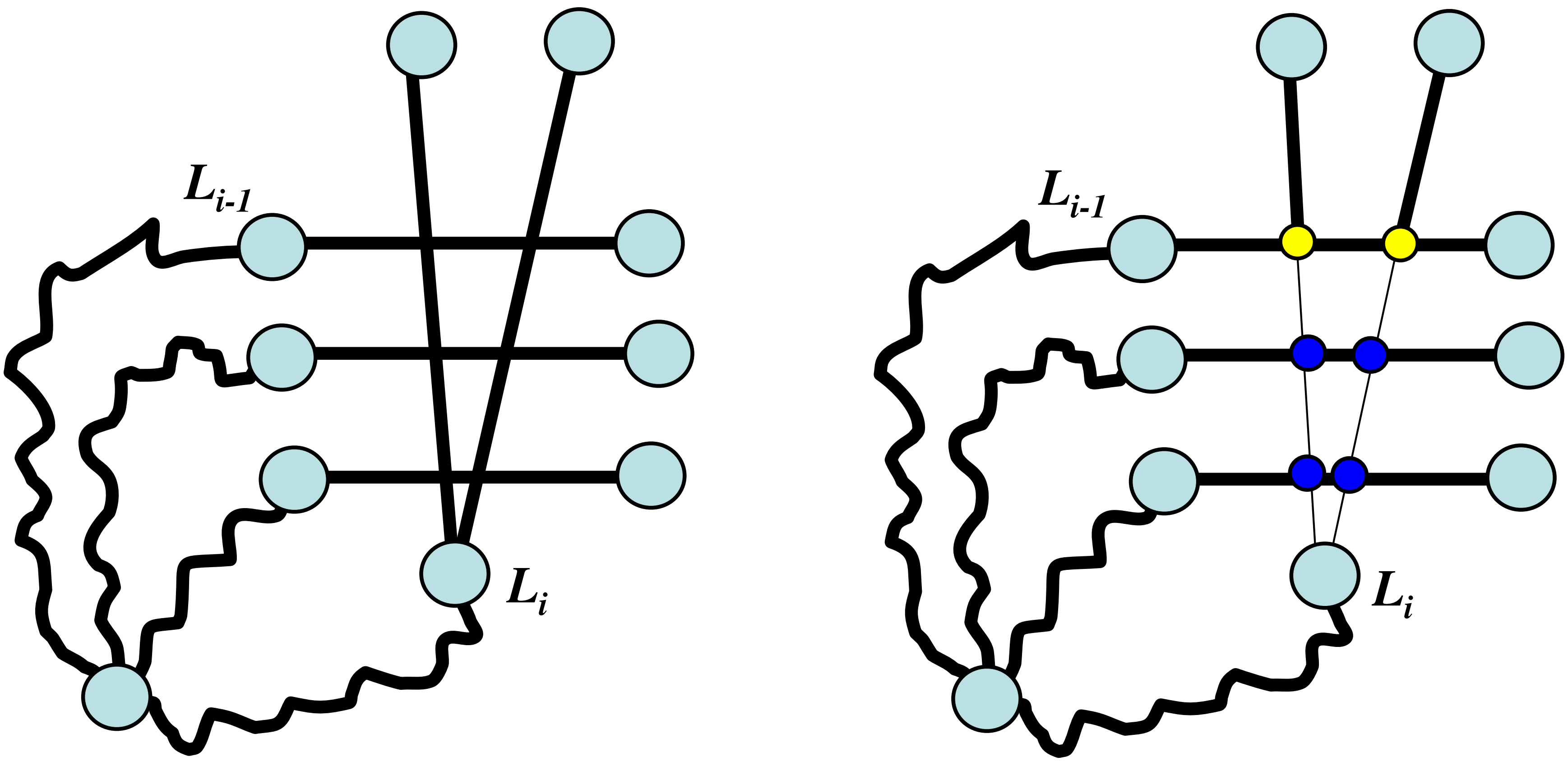


Step 2: Convert the Unit Disk Graph G into a planar graph G_p and T into a spanning tree T_p for G_p

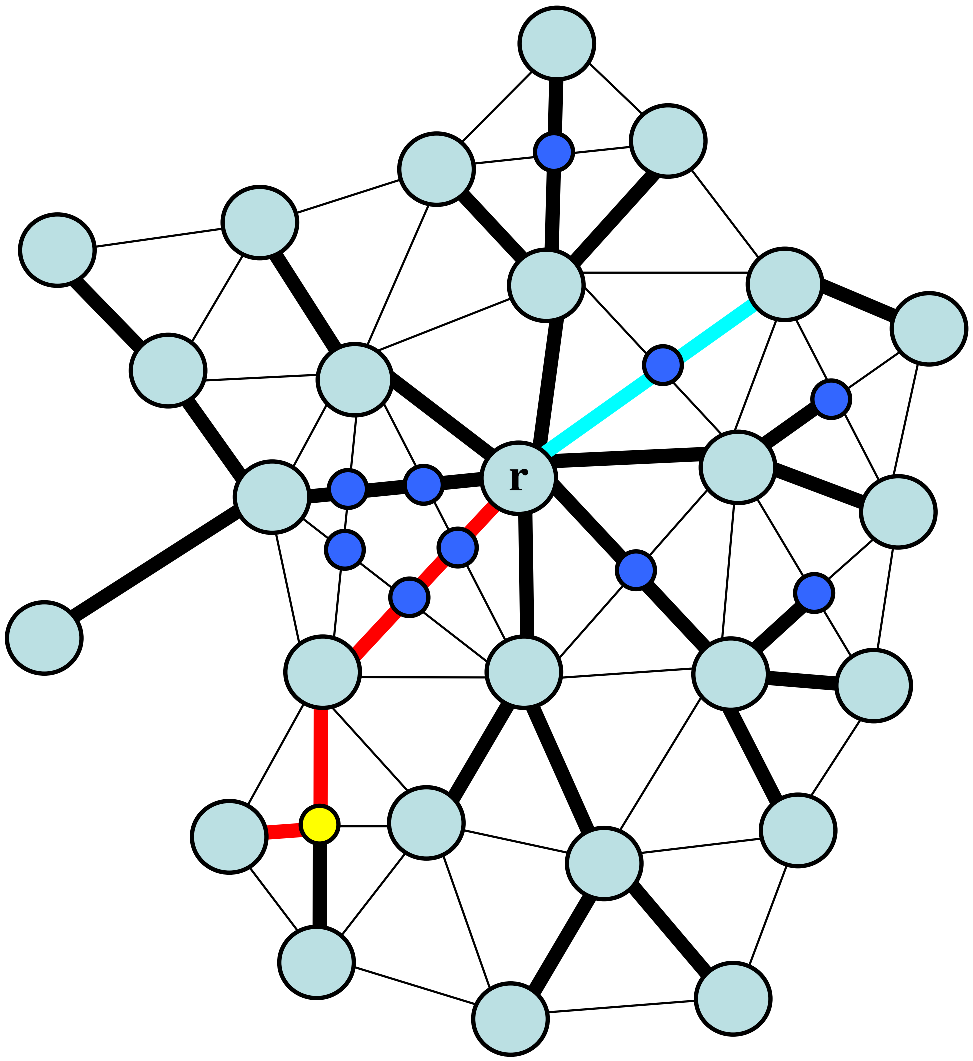


Challenging problem: an edge has multiple intersections in $\mathcal{G}$

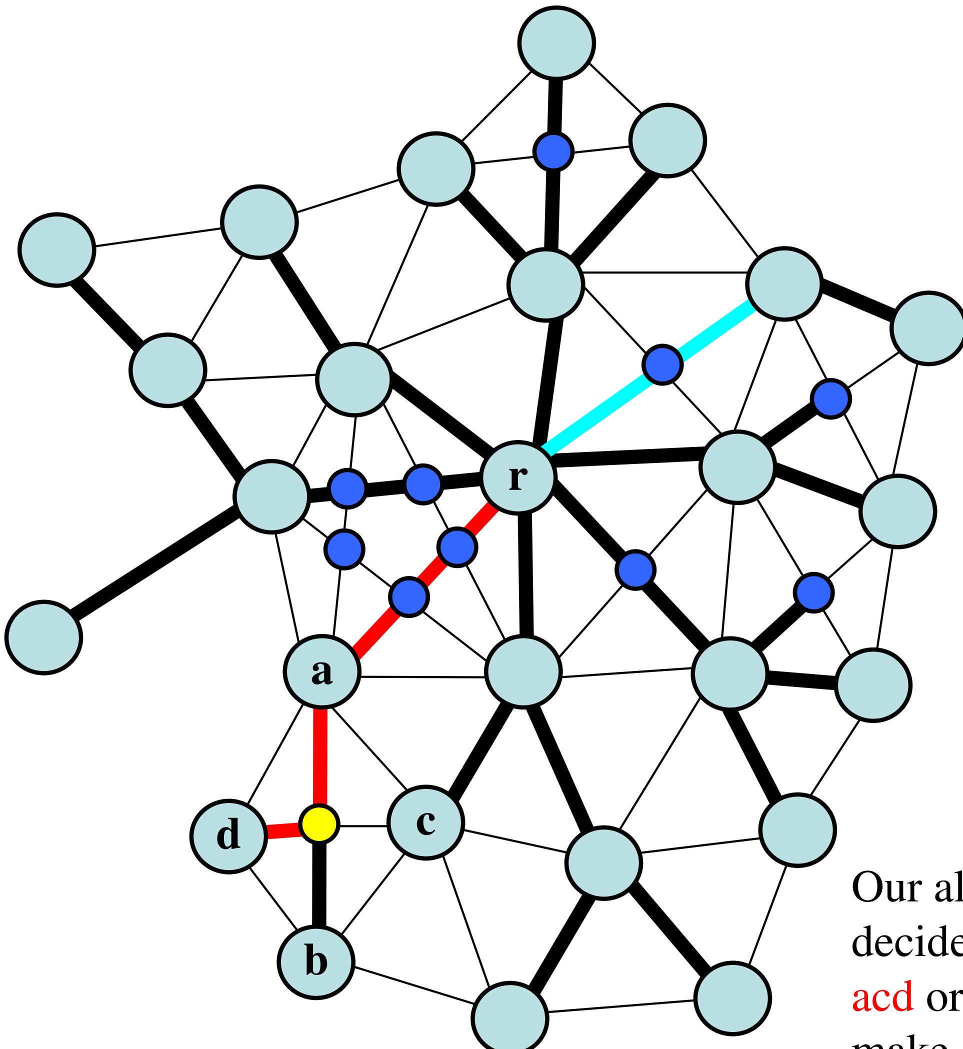
- Our algorithm can deal with this case.
  For Example:

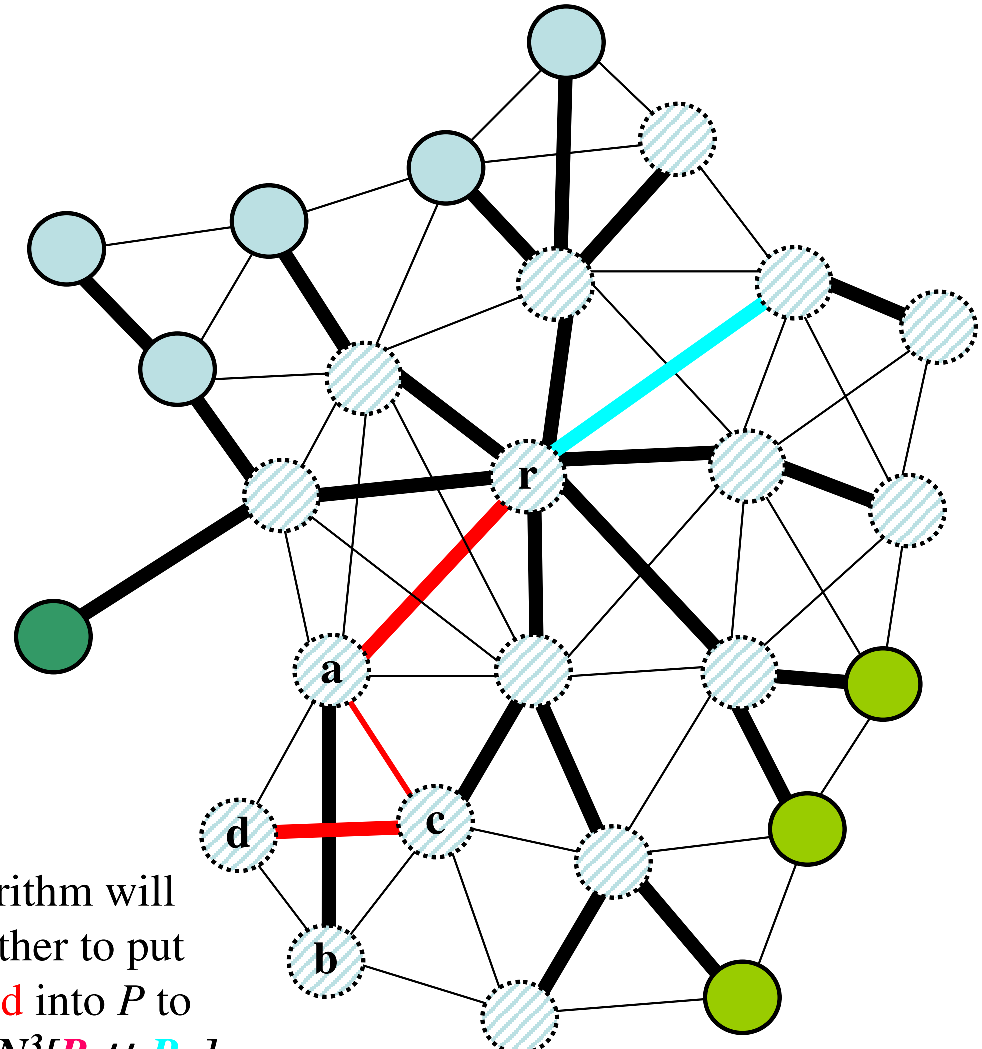


Step 3: Apply Lipton&Tarjan's separator theorem to the planar graph G_p and spanning tree T_p to find a balanced separator S_p for G_p



Step 4: From S_p, reconstruct a balanced separator S for G

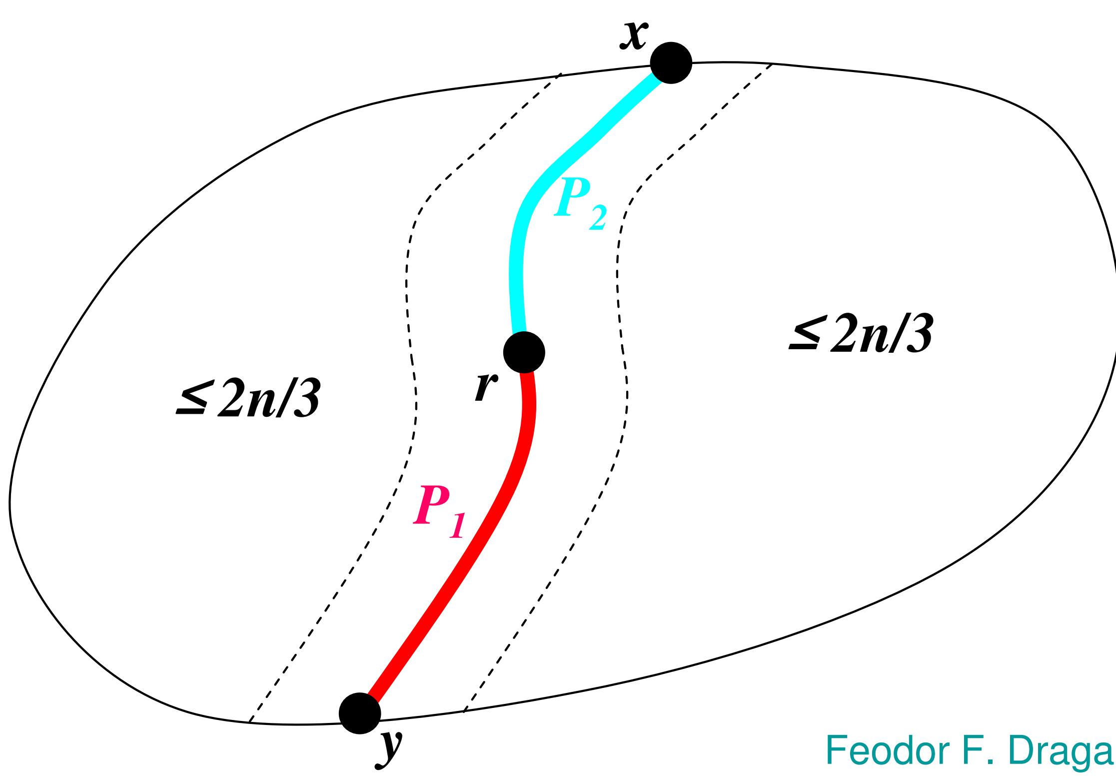




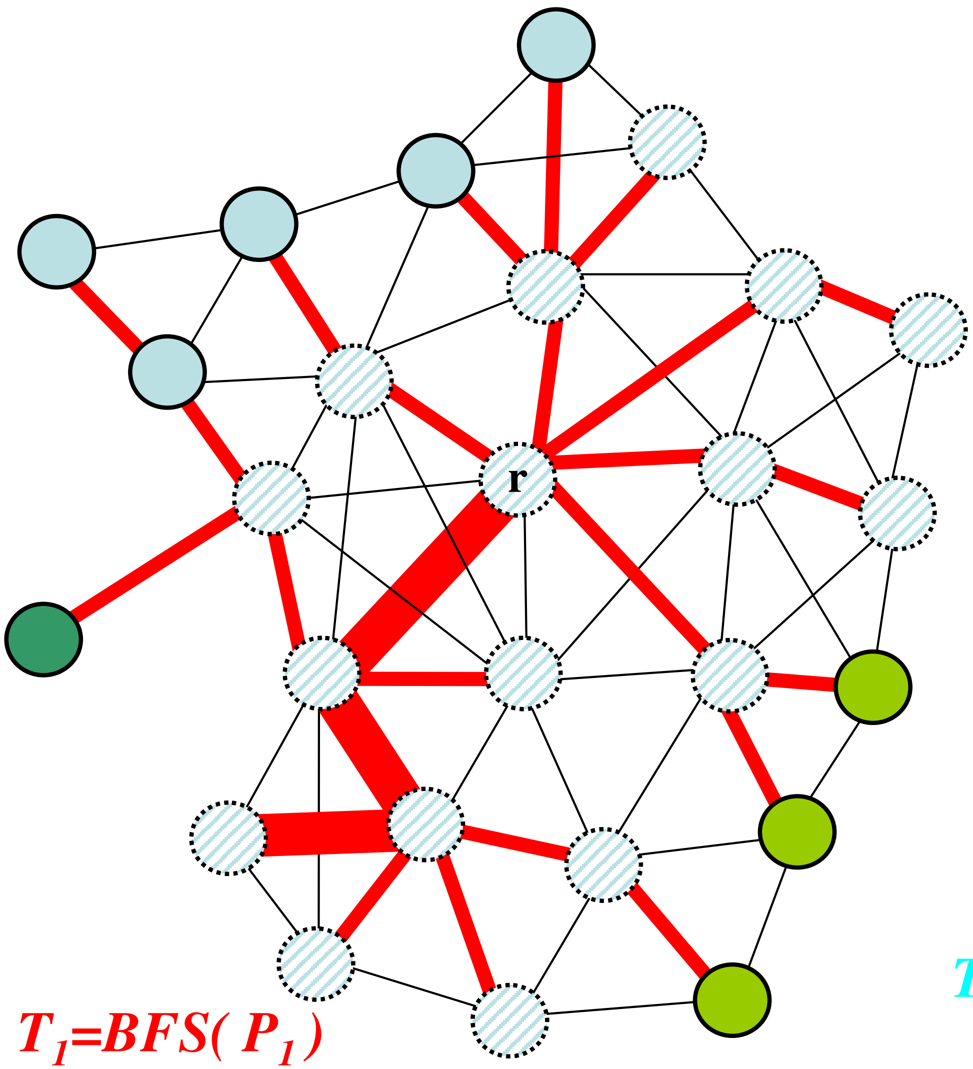
Our algorithm will decide either to put acd or abd into P to make $S=N^3[P_1 \cup P_2]$ a balanced separator.

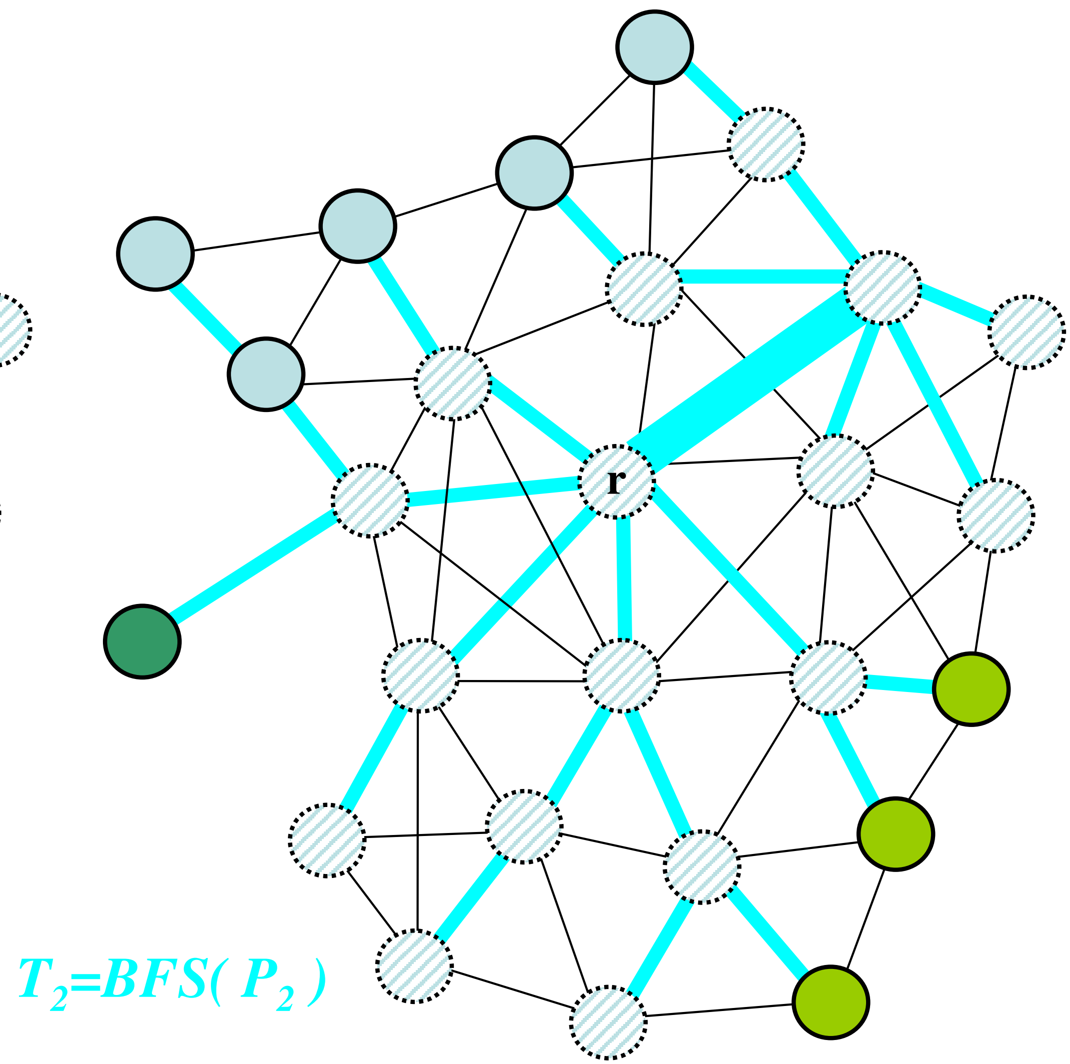
Separator theorem

- $S=N^3_G[P_1 \cup P_2]$ is a balanced separator for G with 2/3-split, i.e., removal of S from G leaves no connected component with more than $2/3n$ vertices



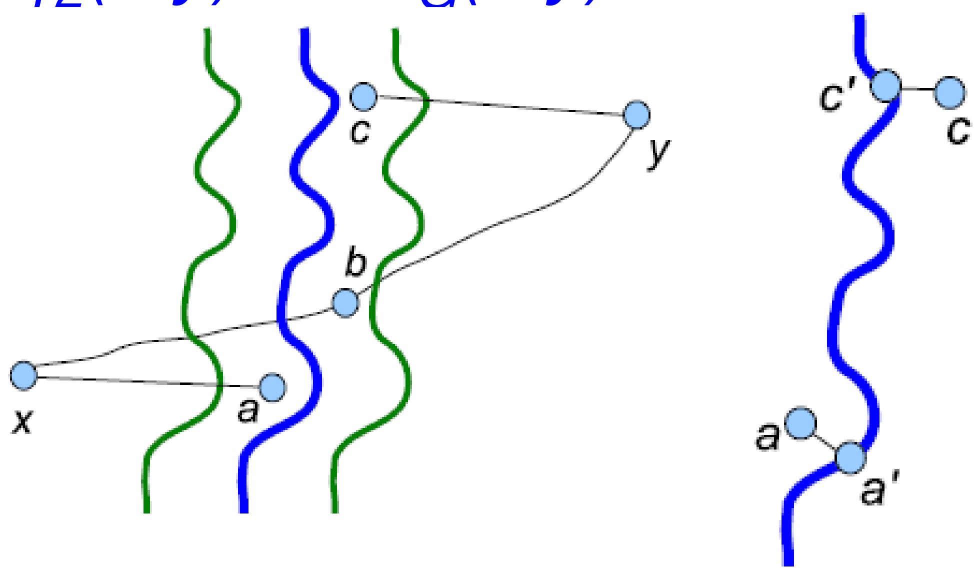
Constructing two spanning trees for a balanced separator





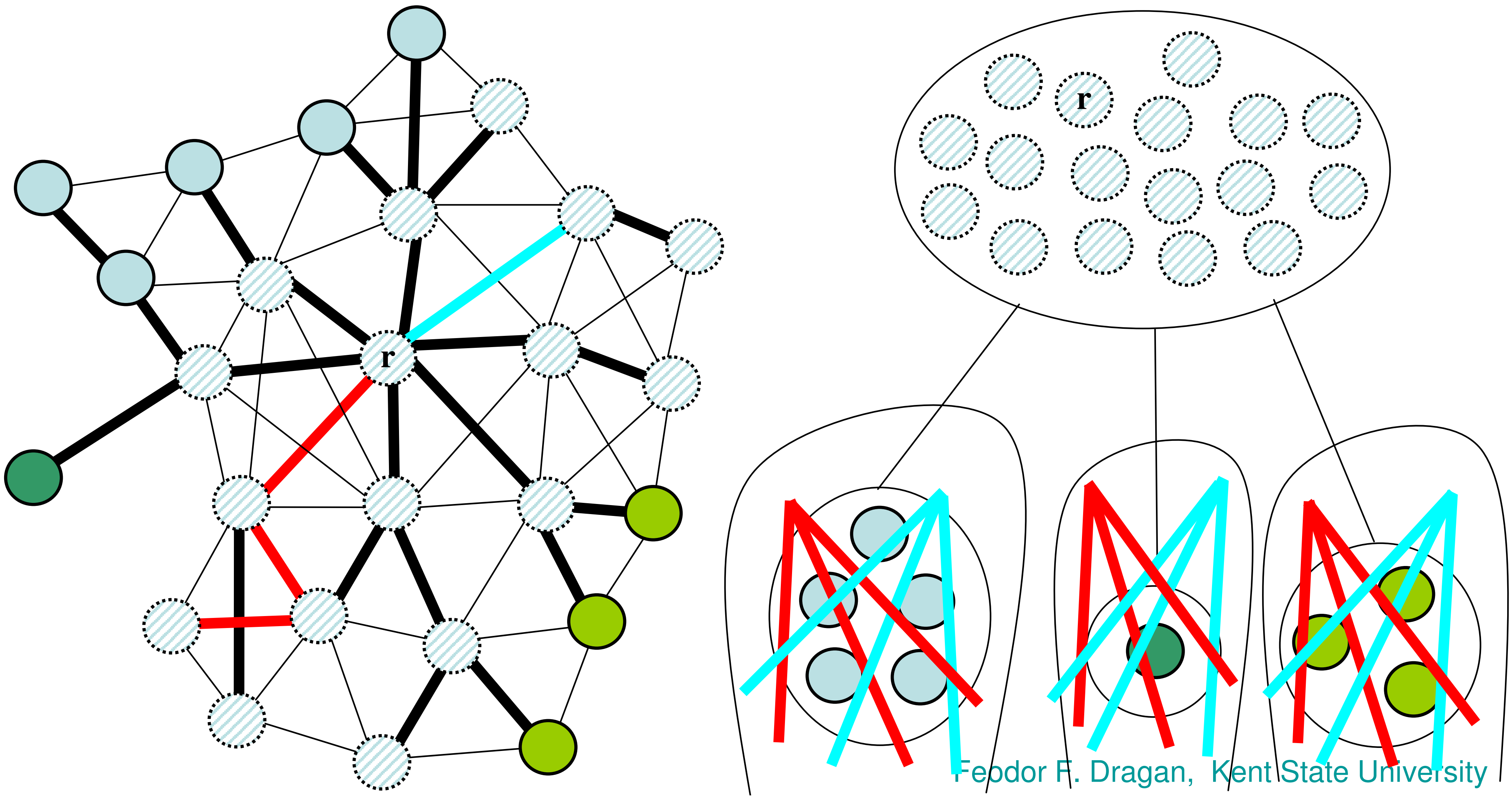
Lemma for the two spanning trees

- Let x, y be two arbitrary vertices of G and $P(x,y)$ be a (hop-) shortest path between x and y in G. If $P(x,y) \cap S \neq \emptyset$, then
  - $d_{T1}(x,y) \leq 3d_G(x,y)+12$ or
  - $d_{T2}(x,y) \leq 3d_G(x,y)+12$



Constructing two spanning trees per level of decomposition

- For each layer of the decomposition tree, construct *local spanning trees (shortest path trees in the subgraph)*



Theorem for collective tree spanners

- Any unit disk graph G with n vertices and m edges admits a system $\mathcal{T}(G)$ of at most $2\log_{3/2}n+2$ collective tree $(3,12)$-spanners, i.e., for any two vertices x and y in G, there exists a spanning tree $T\in \mathcal{T}(G)$ with $d_T(x,y) \leq 3d_G(x,y)+12$

Applications: Distance Labeling Scheme and Routing Labeling Scheme

- **Distance Labeling Scheme**: The family of n-vertex unit disk graphs admits an $O(\log^2 n)$ bit *(3,12)*-approximate distance labeling scheme with $O(\log n)$ time distance decoder.
- **Routing Labeling Scheme**: The family of n-vertex unit disk graphs admits an $O(\log n)$ bit routing labeling scheme. The Scheme has hop *(3,12)*-route-stretch. Once computed by the sender in $O(\log n)$ time, headers never change, and the routing decision is made in constant time per vertex.

Extension to routing labeling scheme with bounded length route-stretch

- The family of *n*-vertex unit disk graphs admits an *O(log n)* bit routing labeling scheme. The scheme has length *(5,13)*-route-stretch. Once computed by the sender in *O(log n)* time, headers never change, and the routing decision is made in constant time per vertex.

Open questions

- Does there exist a distance or a routing labeling scheme that can be locally constructed for Unit Disk Graphs?
- *Does there exist* a balanced separator *of form* $S=N_G[P_1 \cup P_2]$?

Thank You