New Graph Classes of Bounded Clique-Width*

Andreas Brandstädt[1], Feodor F. Dragan[2],
Hoàng-Oanh Le[1], and Raffaele Mosca[3]

[1] Institut für Theoretische Informatik, Fachbereich Informatik,
Universität Rostock, 18051 Rostock, Germany
{ab,hoang-oanh.le}@informatik.uni-rostock.de
[2] Department of Computer Science, Kent State University, Kent, Ohio 44242, USA
dragan@cs.kent.edu
[3] Via Latina 7, Pescara 65121 Italy
ciufolini@tiscalinet.it

Abstract. Clique-width of graphs is a major new concept with respect to efficiency of graph algorithms; it is known that every algorithmic problem expressible in a certain kind of Monadic Second Order Logic called *LinEMSOL*($\tau_{1,L}$) by Courcelle, Makowsky and Rotics, is solvable in linear time on any graph class with bounded clique-width for which a k-expression for the input graph can be constructed in linear time. The concept of clique-width extends the one of treewidth since bounded treewidth implies bounded clique-width.
We give a complete classification of all graph classes defined by forbidden one-vertex extensions of the P_4 with respect to their clique-width. Our results extend and improve recently published structural and complexity results in a systematic way.

1 Introduction

Recently, in connection with graph grammars, Courcelle, Engelfriet and Rozenberg in [15] introduced the concept of clique-width of a graph which has attracted much attention due to the fact that, in [16], Courcelle, Makowsky and Rotics have shown that every graph problem definable in LinEMSOL($\tau_{1,L}$) (a variant of Monadic Second Order Logic) is linear-time solvable on graphs with bounded clique-width if a k-expression describing the input graph is given. The problems Vertex Cover, Maximum Weight Stable Set (MWS), Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems. Note that every class of bounded treewidth has bounded clique-width as well (see [17].

It is known that the class of P_4-free graphs (also called *cographs*) is exactly the class of graphs having clique-width at most 2, and a 2-expression can be found in linear time along the cotree of a cograph. Due to the basic importance of cographs, it is of interest to consider graph classes defined by forbidden one-vertex extensions of a P_4 - see Figure 1 - which are natural generalizations of

* Research of the first author partially supported by Kent State University, Kent, Ohio, research of the third and fourth author partially supported by German Research Community DFG Br 1446-4/1

L. Kučera (Ed.): WG 2002, LNCS 2573, pp. 57–67, 2002.

cographs. The aim of this paper is to investigate the structure and to classify the clique-width of all these graph classes in a systematic way. This is also motivated by known examples such as the (P_5,co-P_5,bull)-free graphs studied by Fouquet in [20] (see Theorem 16) and the (P_5,co-P_5,chair)-free graphs studied by Fouquet and Giakoumakis in [21] (see Theorem 13). Moreover, there are papers such as [29, 34] dealing with (chair,co-P,gem)-free graphs and [24] dealing with (P_5,P,gem)-free graphs where it is shown that the MWS problem can be solved in polynomial time on these classes. Our results imply bounded clique-width and linear time for the MWS problem and any other LinEMSOL($\tau_{1,L}$) definable problem for these classes as well as for many other examples. This also continues research done in [2–5, 8–11].

Due to space limitations, all proofs in this extended abstract are omitted; they are contained in the full version of the paper.

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. The edges between two disjoint vertex sets X, Y form a *join*, denoted by ① (*co-join*, denoted by ⓪) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M. A module is *trivial* if it is either the empty set, a one-vertex set or the entire vertex set V. Nontrivial modules are called *homogeneous sets*. A graph is *prime* if it contains only trivial modules. The notion of modules plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms - see e.g. [32] for modular decomposition of discrete structures and its algorithmic use.

For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. Let co-$G = \overline{G} = (V, \overline{E})$ denote the complement graph of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k - 1$ edges, and for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* is a C_k, $k \geq 5$. Note that the P_4 is the smallest nontrivial prime graph and the complement of a P_4 is a P_4 itself.

See Figure 1 for the definition of the chair, P, bull, gem and their complements. Note that the complement of a bull is a bull itself. The *diamond* is the $K_4 - e$, i.e., a four vertex clique minus one edge.

Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$. There are many papers on the structure and algorithmic use of prime $\mathcal{F}$-free graphs for $\mathcal{F}$ being a set of P_4 extensions; see e.g. [20–22, 25, 26, 28, 2, 4, 3, 11]. A graph is a *split graph* if G is partitionable into a clique and a stable set. It is known [19] that G is a split graph if and only if it is a ($2K_2$,C_4,C_5)-free graph.

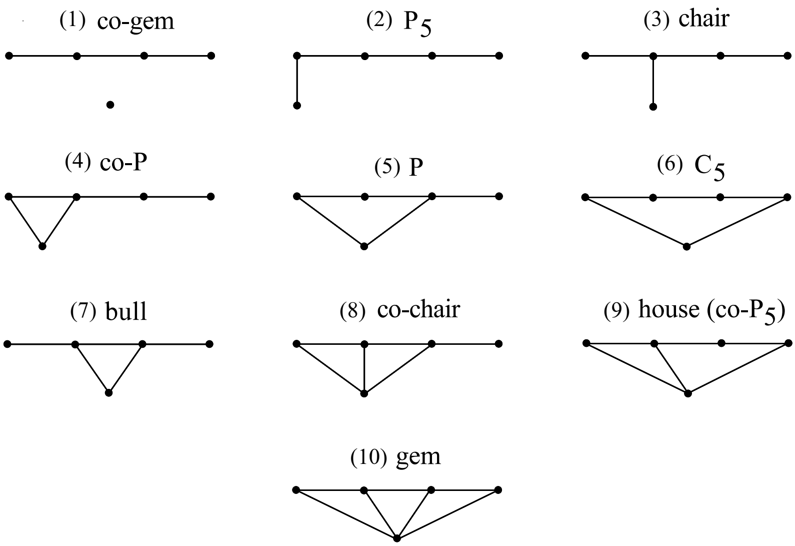

Fig. 1. All one-vertex extensions of a P_4

In what follows, we need the following classes of (prime) graphs:

- G is a *thin spider* if its vertex set is partitionable into a clique C and a stable set S with $|C| = |S|$ or $|C| = |S| + 1$ such that the edges between C and S are a matching and at most one vertex in C is not covered by the matching (an unmatched vertex is called the *head of the spider*).
- A graph is a *thick spider* if it is the complement of a thin spider.
- G is *matched co-bipartite* if its vertex set is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| - 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching.
- G is *co-matched bipartite* if G is the complement graph of a matched co-bipartite graph.
- A bipartite graph $B = (X, Y, E)$ is a *chain graph* [33] if for all vertices from X (Y), their neighborhoods in Y (X) are linearly ordered. If moreover, $|X| = |Y|$ and for all vertices from X (Y), their neighborhoods in Y (X) have size $1, 2, \ldots, |Y|$ ($1, 2, \ldots, |X|$) then these graphs are prime.
- G is a *co-bipartite chain graph* if it is the complement of a bipartite chain graph.
- G is an *enhanced co-bipartite chain graph* if it is partitionable into a co-bipartite chain graph with cliques C_1, C_2 and three additional vertices a, b, c (a and c optional) such that $N(a) = C_1 \cup C_2$, $N(b) = C_1$, and $N(c) = C_2$, and there are no other edges in G.
- G is an *enhanced bipartite chain graph* if it is the complement of an enhanced co-bipartite chain graph.

2 Cographs, Clique-Width and Expressibility of Problems

The P_4-free graphs (also called *cographs*) play a fundamental role in graph decomposition; see [14] for linear time recognition of cographs, [12–14] for more information on P_4-free graphs and [7] for a survey on this graph class and related ones. For a cograph G, either G or its complement is disconnected, and the *cotree* of G expresses how the graph is recursively generated from single vertices by repeatedly applying join and co-join operations. The cotree representation allows to solve various NP-hard problems in linear time when restricted to cographs, among them the problems Maximum Weight Stable Set and Maximum Weight Clique. Note that the cographs are those graphs whose modular decomposition tree contains only join and co-join nodes as internal nodes.

Based on the following operations on vertex-labeled graphs, namely

– creation of a vertex labeled by integer l,
– disjoint union (i.e., co-join),
– join between all vertices with label i and all vertices with label j for $i \neq j$, and
– relabeling vertices of label i by label j,

the notion of *clique-width* $cwd(G)$ of a graph G is defined in [15] as the minimum number of labels which are necessary to generate G by using these operations. Cographs are exactly the graphs whose clique-width is at most two.

A *k-expression* for a graph G of clique-width k describes the recursive generation of G by repeatedly applying these operations using at most k different labels.

Proposition 1 ([16, 17]) *The clique-width of a graph G is the maximum of the clique-width of its prime subgraphs, and the clique-width of the complement graph $\overline{G}$ is at most twice the clique-width of G.*

Recently, the concept of clique-width of a graph attracted much attention since it gives a unified approach to the efficient solution of many algorithmic graph problems on graph classes of bounded clique-width via the expressibility of the problems in terms of logical expressions.

In [16], it is shown that every problem definable in a certain kind of Monadic Second Order Logic, called *LinEMSOL($\tau_{1,L}$)* in [16], is linear-time solvable on any graph class with bounded clique-width for which a k-expression can be constructed in linear time.

Hereby, in [16], it is mentioned that, roughly speaking, MSOL(τ_1) is Monadic Second Order Logic with quantification over subsets of vertices but not of edges; MSOL($\tau_{1,L}$) is the restriction of MSOL(τ_1) with the addition of labels added to the vertices, and LinEMSOL($\tau_{1,L}$) is the restriction of MSOL($\tau_{1,L}$) which allows to search for sets of vertices which are optimal with respect to some linear evaluation functions.

The problems Vertex Cover, Maximum Weight Stable Set, Maximum Weight Clique, Steiner Tree and Domination are examples of LinEMSOL($\tau_{1,L}$) definable problems.

Theorem 1 ([16]) *Let $\mathcal{C}$ be a class of graphs of clique-width at most k such that there is an $\mathcal{O}(f(|E|,|V|))$ algorithm, which for each graph G in $\mathcal{C}$, constructs a k-expression defining it. Then for every LinEMSOL($\tau_{1,L}$) problem on $\mathcal{C}$, there is an algorithm solving this problem in time $\mathcal{O}(f(|E|,|V|))$.*

As an application, it was shown in [16] that P_4-sparse graphs and some variants of them have bounded clique-width. Hereby, a graph is *P_4-sparse* if no set of five vertices in G induces at least two distinct P_4's [25, 26]. From the definition, it is obvious that a graph is P_4-sparse if and only if it contains no C_5, P_5, $\overline{P_5}$, P, $\overline{P}$, chair, co-chair (see Figure 1). See [11] for a systematic investigation of superclasses of P_4-sparse graphs.

In [25], it was shown that the prime P_4-sparse graphs are the spiders (which were called *turtles* in [25]), and according to Proposition 1 and the fact that the clique-width of thin spiders is at most 4 (which is easy to see), it follows that P_4-sparse graphs have bounded clique-width.

Recently, variants of P_4-sparse graphs attracted much attention because of their applications in areas such as scheduling, clustering and computational semantics. Moreover, all these classes are natural generalizations of cographs.

It is straightforward to see that the clique-width of matched co-bipartite (co-matched bipartite) graphs, bipartite chain (co-bipartite chain) graphs as well as the clique-width of induced paths and cycles is at most 4, and corresponding k-expressions can be determined in linear time. Distance-hereditary graphs are the (house,hole,domino,gem)-free graphs - see [1, 7]. In [23], Golumbic and Rotics have shown that their clique width is at most 3 and corresponding k-expressions can be determined in linear time.

3 Further Tools

Lemma 1 ([27]) *If a prime graph contains an induced C_4 (induced $2K_2$) then it contains an induced co-P_5 or A or domino (induced P_5 or co-A or co-domino).*

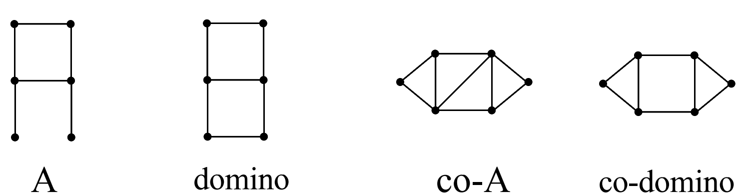

Fig. 2. The A and domino and their complements

The proof of Lemma 1 can be extended in a straightforward way to the case of a diamond instead of a C_4. For this purpose let us call *d-A* the graph resulting from an A graph by adding an additional diagonal edge in the C_4, and *d-domino*

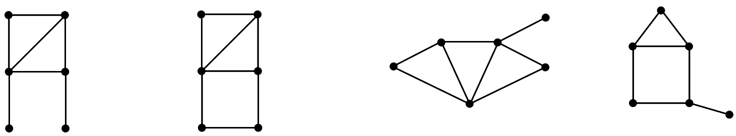

Fig. 3. The *d-A* and *d*-domino and their complements

the graph resulting from a domino graph by adding an additional diagonal edge in one of the C_4's - see Figure 3.

Lemma 2 *If a prime graph contains an induced diamond (co-diamond) then it contains an induced gem or d-A or d-domino (co-gem or co-d-A or co-d-domino).*

Theorem 2 ([2]) *Prime (P_5,diamond)-free graphs are either matched co-bipartite or a thin spider or an enhanced bipartite chain graph or have at most 9 vertices.*

For a structure description of (P_5,gem)-free graphs see [4] where the following Lemma is shown:

Lemma 3 ([4]) *Prime (P_5,gem)-free graphs containing a co-domino are matched co-bipartite.*

Theorem 3 ([9]) *If G is a prime (diamond,co-diamond)-free graph then G or $\overline{G}$ is either a matched co-bipartite graph or G has at most 9 vertices.*

Lemma 4 ([3]) *Prime chair-free bipartite graphs are co-matched bipartite, a path, or a cycle.*

Lemma 5 ([10]) *Prime chair-free split graphs are spiders.*

Lemma 6 ([3, 18]) *Prime (bull, chair)-free graphs containing a co-diamond are either co-matched bipartite or a cycle or a path.*

Lemma 7 *Prime co-gem-free bipartite graphs are co-matched bipartite.*

Lemma 8 *Prime (co-diamond,gem)-free graphs containing a diamond have at most 11 vertices.*

4 Structure and Clique-Width Results

Figure 4 contains all combinations of three forbidden P_4 extensions (enumerated according to Figure 1). Each class together with its complement class occurs only once; we take the lexicographically smaller class; for example, the (P_5,co-P_5,gem)-free graphs are the (2,9,10)-free graphs, and its complement class is the class of (P_5,co-P_5,co-gem)-free graphs, i.e., the (1,2,9)-free graphs; in Figure 4, only the class (1,2,9) occurs.

1 2 3 −	1 2 4 −	1 2 5 −	1 2 6 −	1 2 7 −	1 2 8 −	1 2 9 +	1 2 10 +
1 3 4 −	1 3 5 −	1 3 6 −	1 3 7 −	1 3 8 −	1 3 9 +	1 3 10 +	1 4 5 −
1 4 6 −	1 4 7 −	1 4 8 −	1 4 9 +	1 4 10 +	1 5 6 −	1 5 7 −	1 5 8 +
1 5 9 +	1 6 7 −	1 6 8 −	1 6 9 +	1 6 10 +	1 7 8 +	1 7 9 +	1 7 10 +
1 8 9 +	2 3 4 −	2 3 5 −	2 3 6 −	2 3 7 −	2 3 8 −	2 3 9 +	2 4 5 −
2 4 6 −	2 4 7 −	2 4 8 −	2 4 9 −	2 5 6 −	2 5 7 −	2 5 8 +	2 6 7 −
2 6 8 −	2 6 9 −	2 7 8 +	2 7 9 +	3 4 5 −	3 4 6 −	3 4 7 −	3 4 8 −
3 5 6 −	3 5 7 −	3 6 7 −	3 6 8 −	3 7 8 +	4 5 6 −	4 5 7 −	4 6 7 −

Fig. 4. All combinations of three forbidden P_4 extensions; + (−) denotes bounded (unbounded) clique-width

Theorem 4 ((1,2,9)) *If G is a prime (P_5,co-P_5,gem)-free graph then G is distance-hereditary or a C_5.*

The subsequent Theorems 5 and 6 are a simple consequence of Lemma 2.

Theorem 5 ((1,2,10),(1,4,10)) *If G is a prime (P_5,gem,co-gem)-free or (gem, co-P, co-gem)-free graph then G is (diamond,co-diamond)-free.*

Theorem 6 ((1,3,9),(1,7,9),(1,8,9)) *If G is a prime (P_5,gem,co-chair)-free or (P_5,gem,bull)-free or (P_5,gem,chair)-free graph then G is (P_5,diamond)-free.*

Thus, the structure of the classes considered in Theorems 5 and 6 is described in Theorems 2, 3 respectively.

Theorem 7 ((1,3,10)) *If G is a prime (co-gem,chair,gem)-free graph then G or $\overline{G}$ is a matched co-bipartite graph or G has at most 11 vertices.*

Theorem 8 ((1,4,9)) *If G is a prime (P_5,P,gem)-free graph then G is a matched co-bipartite graph or a distance-hereditary graph or a C_5 .*

Theorem 9 ((1,5,8), [8]) *If G is a prime (chair, co-P, gem)-free graph then G fulfills one of the following conditions:*

(*i*) *G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*
(*ii*) *G is a thin spider;*
(*iii*) *G is a co-matched bipartite graph;*
(*iv*) *G has at most 11 vertices.*

The next theorem is partially based on the structure of (P_5, gem)-free graphs described in [4]:

Theorem 10 ([5]) *The clique width of (P_5, gem)-free graphs is at most 9.*

Thus, according to Theorem 10, the classes (1,5,9) and (1,6,9) have bounded clique-width as well; however, their structure is more complicated than the previous examples and we do not know any linear time algorithm for determining k-expressions for these graphs.

Theorem 11 ([6]) *The clique-width of (gem,co-gem)-free graphs is at most 24.*

The proof of Theorem 11 is technically very involved and does not give any simpler structure description for the subclasses (1,6,10), (1,7,10), (1,6,7,10); we do not know any linear time algorithm for determining k-expressions for these graphs.

Theorem 12 ((1,7,8)) *If G is a prime (bull,chair,gem)-free graph then G fulfills one of the following conditions:*

(*i*) *G or $\overline{G}$ is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*
(*ii*) *G or $\overline{G}$ is a co-matched bipartite graph;*
(*iii*) *G has at most 11 vertices.*

Theorem 13 ((2,3,9), [21]) *If G is a prime (P_5,$\overline{P_5}$,chair)-free graph then G is either a co-bipartite chain graph or a spider or C_5.*

Theorem 14 ((2,5,8), [11]) *If G is a prime (chair,co-P,house)-free graph then G fulfills one of the following conditions:*

(*i*) *G is an induced path P_k, $k \geq 4$, or an induced cycle C_k, $k \geq 5$;*
(*ii*) *G is a co-matched bipartite graph;*
(*iii*) *G is a spider.*

Theorem 15 ((2,7,8), [3]) *If G is a prime ($\overline{P_5}$,bull,chair)-free graph then G is either a co-matched bipartite graph or an induced path or cycle or $\overline{G}$ is (P_5,diamond)-free.*

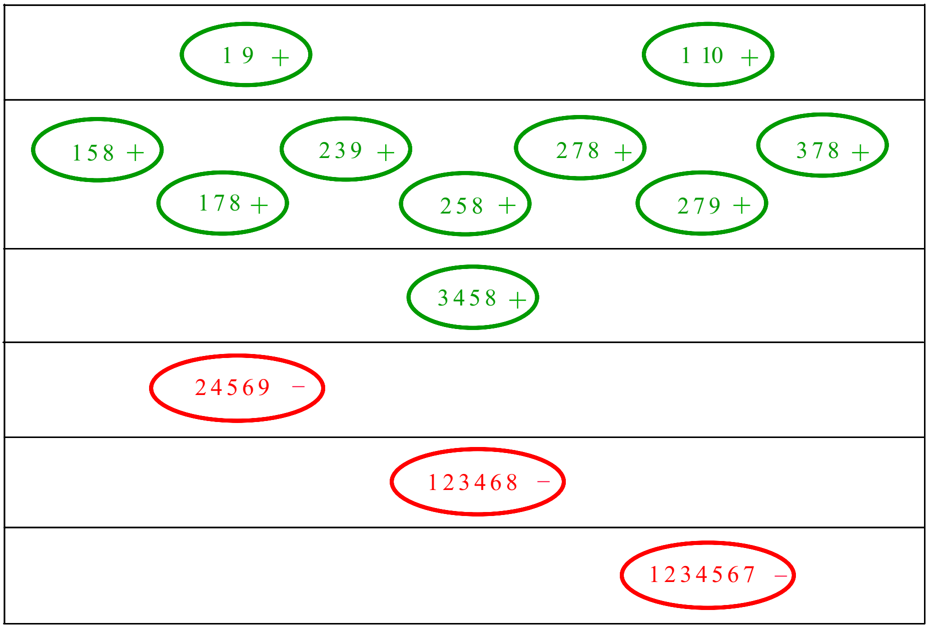

Fig. 5. Essential classes for all combinations of forbidden 1-vertex P_4 extensions; + (−) denotes bounded (unbounded) clique-width

Theorem 16 ((2,7,9), [20]) *If G is a prime (P_5,$\overline{P_5}$,bull)-free graph then G or $\overline{G}$ is a bipartite chain graph or a C_5.*

Theorem 17 ((3,7,8), [3]) *If G is a prime (bull,chair,co-chair)-free graph then G or $\overline{G}$ is either a co-matched bipartite graph or an induced path or cycle.*

Corollary 1 *Every LinEMSOL($\tau_{1,L}$) definable problem is solvable in linear time on all graph classes of bounded clique-width (i.e. not indicated with − in Figure 4), except the classes (1,5,9), (1,6,9), (1,6,10), (1,7,10), (1,6,7,10).*

Makowsky and Rotics have shown in [31] that the following grid types have unbounded clique-width:

- the F_n grid (whose complements are (1,2,3,4,6,8)-free);
- the $H_{n,q}$ grid (whose complements are (1,2,3,4,5,6,7)-free)

Moreover, they show that split graphs have unbounded clique width. This implies unbounded clique-width for all classes with − in Figure 4 and in Figure 5.

Let $\mathcal{F}$ denote the 10 one-vertex extensions of the P_4 (see Figure 1). For $\mathcal{F}' \subseteq \mathcal{F}$, there are 1024 classes of $\mathcal{F}'$-free graphs. Figure 5 shows all inclusion-minimal classes of unbounded clique-width and all inclusion-maximal classes of bounded clique-width. As before, we consider the class of $\mathcal{F}'$-free graphs together with its complement class, the co-$\mathcal{F}'$-free graphs, and mention only the lexicographically smaller class. Note that any subclass of bounded clique-width has bounded clique-width as well, whereas any superclass of unbounded clique-width has unbounded clique-width as well. Obviously, a graph with at least 5 vertices is a cograph if and only if it contains none of the 10 possible one-vertex extensions of a P_4. For $|\mathcal{F}'| \in \{9, 8\}$, all these classes have bounded clique-width. For $|\mathcal{F}'| = 7$ there is exactly one inclusion-minimal class (together with its complement class) of unbounded clique-width namely (1,2,3,4,5,6,7) (enumeration with respect to Figure 1), and similarly for $|\mathcal{F}'| = 6$ and $|\mathcal{F}'| = 5$. For $|\mathcal{F}'| = 4$ there is exactly one inclusion-maximal class of bounded clique-width namely (3,4,5,8). For $|\mathcal{F}'| = 3$, the inclusion-maximal classes of bounded clique-width are (1,5,8), (1,7,8), (2,3,9), (2,5,8), (2,7,8), (2,7,9), (3,7,8). For $|\mathcal{F}'| = 2$, the only classes of bounded clique-width are (1,9) and (1,10), and for $|\mathcal{F}'| = 1$, all classes have unbounded clique-width.

Open problem.

1. Is there a linear time algorithm for determining a k-expression with constant k for the classes (1,9), (1,5,9), (1,6,9), (1,10), (1,6,10), (1,7,10), (1,6,7,10)?

Acknowledgement

The authors thank Van Bang Le for helpful discussions.

References

1. H.-J. Bandelt, H.M. Mulder, Distance-hereditary graphs, *J. Combin. Theory (B)* 41 (1986) 182-208
2. A. Brandstädt, (P_5,diamond)-Free Graphs Revisited: Structure, Bounded clique-width and Linear Time Optimization, *Manuscript* 2000; accepted for *Discrete Applied Math.*
3. A. Brandstädt, C.T. Hoàng, V.B. Le, Stability Number of Bull- and Chair-Free Graphs Revisited, *Manuscript* 2001; accepted for *Discrete Applied Math.*
4. A. Brandstädt, D. Kratsch, On the structure of (P_5,gem)-free graphs, *Manuscript* 2001
5. A. Brandstädt, H.-O. Le, R. Mosca, Chordal co-gem-free graphs have bounded clique-width, *Manuscript* 2002
6. A. Brandstädt, H.-O. Le, R. Mosca, (Gem,co-gem)-free graphs have bounded clique-width, *Manuscript* 2002
7. A. Brandstädt, V.B. Le, J. Spinrad, Graph Classes: A Survey, *SIAM Monographs on Discrete Math. Appl., Vol.* 3, SIAM, Philadelphia (1999)
8. A. Brandstädt, H.-O. Le, J.-M. Vanherpe, Structure and Stability Number of (Chair, Co-P, Gem)-Free Graphs, *Manuscript* 2001
9. A. Brandstädt, S. Mahfud, Linear time for Maximum Weight Stable Set on (claw,co-claw)-free graphs and similar graph classes, *Manuscript* 2001; to appear in *Information Processing Letters*
10. A. Brandstädt, R. Mosca, On the Structure and Stability Number of P_5- and Co-Chair-Free Graphs, *Manuscript* 2001; accepted for *Discrete Applied Math.*
11. A. Brandstädt, R. Mosca, On Variations of P_4-Sparse Graphs, *Manuscript* 2001

12. D.G. Corneil, H. Lerchs, L. Stewart-Burlingham, Complement reducible graphs, *Discrete Applied Math.* 3 (1981) 163-174
13. D.G. Corneil, Y. Perl, L.K. Stewart, Cographs: recognition, applications, and algorithms, *Congressus Numer.* 43 (1984) 249-258
14. D.G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Computing* 14 (1985) 926-934
15. B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, *J. Comput. Syst. Sciences*, 46 (1993) 218-270
16. B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique width, extended abstract in: *Conf. Proc. WG'98*, LNCS 1517 (1998) 1-16; *Theory of Computing Systems* 33 (2000) 125-150
17. B. Courcelle, S. Olariu, Upper bounds to the clique-width of graphs, *Discrete Appl. Math.* 101 (2000) 77-114
18. C. De Simone, On the vertex packing problem, *Graphs and Combinatorics* 9 (1993) 19-30
19. S. Földes, P.L. Hammer, Split graphs, *Congress. Numer.* 19 (1977), 311-315
20. J.-L. Fouquet, A decomposition for a class of $(P_5, \overline{P_5})$-free graphs, *Discrete Math.* 121 (1993) 75-83
21. J.-L. Fouquet, V. Giakoumakis On semi-P_4-sparse graphs, *Discrete Math.* 165-166 (1997) 267-290
22. J.-L. Fouquet, V. Giakoumakis, H. Thuillier, F. Maire, On graphs without P_5 and $\overline{P_5}$, *Discrete Math.* 146 (1995) 33-44
23. M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, *Int. Journal of Foundations of Computer Science* 11 (2000) 423-443
24. A. Hertz, On a graph transformation which preserves the stability number, *Yugoslav Journal of Oper. Res.*, to appear,
25. C.T. Hoàng, *A Class of Perfect Graphs*, *Ms. Sc. Thesis*, School of Computer Science, McGill University, Montreal (1983)
26. C.T. Hoàng, *Perfect Graphs*, *Ph. D. Thesis*, School of Computer Science, McGill University, Montreal (1985)
27. C.T. Hoàng, B. Reed, Some classes of perfectly orderable graphs, *J. Graph Theory* 13 (1989) 445-463
28. B. Jamison, S. Olariu, A unique tree representation for P_4-sparse graphs, *Discrete Appl. Math.* 35 (1992), 115-129
29. V.V. Lozin, Conic reduction of graphs for the stable set problem, *Discrete Math.* 222 (2000) 199-211
30. N.V.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, *Annals of Discrete Mathematics* 56 (1995)
31. J.A. Makowsky, U. Rotics, On the clique-width of graphs with few P_4's, *Int. J. of Foundations of Computer Science* 3 (1999) 329-348
32. R.H. Möhring, F.J. Radermacher, Substitution decomposition for discrete structures and connections with combinatorial optimization, *Annals of Discrete Math.* 19 (1984) 257-356
33. M. Yannakakis, The complexity of the partial order dimension problem, *SIAM J. Algebraic and Discrete Methods* 3 (1982) 351-358
34. I.E. Zverovich, I.I. Zverovich, Extended $(P_5,\overline{P_5})$-free graphs, *Rutcor Research Report* RRR 22-2001 (2001) `http://rutcor.rutgers.edu/~rrr`