Tree-Like Structures in Graphs: A Metric Point of View*

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Abstract. Recent empirical and theoretical work has suggested that many real-life complex networks and graphs arising in Internet applications, in biological and social sciences, in chemistry and physics have tree-like structures from a metric point of view. A number of graph parameters trying to capture this phenomenon and to measure these tree-like structures were proposed; most notable ones being the *tree-stretch*, *tree-distortion*, *tree-length*, *tree-breadth*, Gromov's *hyperbolicity* of a graph, and *cluster-diameter* and *cluster-radius* in a *layering partition* of a graph. If such a parameter is bounded by a constant on graphs then many optimization problems can be efficiently solved or approximated for such graphs. We discuss these parameters and recently established relationships between them for unweighted and undirected graphs; it turns out that all these parameters are at most constant or logarithmic factors apart from each other. We give inequalities describing their relationships and discuss consequences for some optimization problems.

Recent empirical and theoretical work has suggested that many real-life complex networks and graphs arising in Internet applications, in biological and social sciences, in chemistry and physics have tree-like structures from a metric point of view. A number of graph parameters trying to capture this phenomenon and to measure these tree-like structures were proposed; most notable ones being the *tree-stretch* and the *tree-distortion* of a graph, the *tree-length* and the *tree-breadth* of a graph, the Gromov's *hyperbolicity* of a graph, the *cluster-diameter* and the *cluster-radius* in a *layering partition* of a graph.

The *tree-stretch* $\mathsf{ts}(G)$ of a graph $G = (V, E)$ is the smallest number t such that G admits a spanning tree $T = (V, U)$ with $d_T(x, y) \leq t \cdot d_G(x, y)$ for every $x, y \in V$. The *tree-distortion* $\mathsf{td}(G)$ of a graph $G = (V, E)$ is the smallest number α such that G admits a (not necessarily spanning, possibly weighted and having Steiter points) tree $T = (V \cup S, U)$ with $d_G(x, y) \leq d_T(x, y) \leq \alpha \cdot d_G(x, y)$ for every $x, y \in V$. The *tree-length* $\mathsf{tl}(G)$ (resp., *tree-breadth* $\mathsf{tb}(G)$) of a graph G is the smallest number λ such that G admits a Robertson-Seymour's tree-decomposition with bags of diameter (resp., radius) at most λ in G. A graph G is δ-hyperbolic if for any four vertices u, v, w, x, the two larger of the distance

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sums $d(u,v)+d(w,x), d(u,w)+d(v,x), d(u,x)+d(v,w)$ differ by at most 2δ. The hyperbolicity hb(G) of a graph G is the smallest number δ such that G is δ-hyperbolic.

A *layering* of a graph $G=(V,E)$ with respect to a start vertex s is the decomposition of V into the *spheres* $L^i=\{u\in V : d(s,u)=i\}$, $i=0,1,2,\dots,r$. A *layering partition* $\mathcal{L}P(s)=\{L^i_1,\dots,L^i_{p_i} : i=0,1,2,\dots,r\}$ of G is a partition of each L^i into *clusters* $L^i_1,\dots,L^i_{p_i}$ such that two vertices $u,v\in L^i$ belong to the same cluster L^i_j if and only if they can be connected by a path outside the ball $B_{i-1}(s)$ of radius $i-1$ centered at s. The *cluster-diameter* $\Delta_s(G)$ and the *cluster-radius* $R_s(G)$ in a *layering partition* $\mathcal{L}P(s)$ (with respect to s) of a graph G are defined as follows: $R_s(G)$ is the smallest number r such that for any cluster $C\in\mathcal{L}P(s)$ there is a vertex $v\in V$ with $C\subseteq B_r(v)$; $\Delta_s(G):=\max\{d_G(x,y) : x,y$ belong to the same cluster of $\mathcal{L}P(s)\}$.

Each of these graph parameters provides a measure of how close metrically a given graph is to a tree. If such a parameter is bounded by a constant on a graph G, then many optimization problems on G can be solved or approximated efficiently. Note that for a tree T, $\mathsf{ts}(T)=\mathsf{td}(T)=1$, $\mathsf{tl}(T)=\mathsf{tb}(T)=$ *length of the longest edge in* T, and $\mathsf{hb}(T)=R_s(G)=\Delta_s(G)=0$ (i.e., for trees, one has the smallest possible values for those parameters).

In this talk, we discuss these parameters and recently established relationships between them for unweighted and undirected graphs. It turns out that all these parameters are at most constant or logarithmic factors apart from each other. In particular, the following inequalities hold for any n-vertex unweighted and undirected graph $G=(V,E)$:

1) $\mathsf{tb}(G)\le\mathsf{tl}(G)\le 2\cdot\mathsf{tb}(G)$, $R_s(G)\le\Delta_s(G)\le 2\cdot R_s(G)$ ([folklore]);
2) $\mathsf{hb}(G)\le\mathsf{tl}(G)\le O(\mathsf{hb}(G)\cdot\log n)$ and $\mathsf{hb}(G)\le\Delta_s(G)\le O(\mathsf{hb}(G)\cdot\log n)$ ([4,5]);
3) $\mathsf{ts}(G)\ge\mathsf{td}(G)\ge\frac{1}{3}\Delta_s(G)$ and $\mathsf{td}(G)\le 2\cdot\Delta_s(G)+2$ for every $s\in V$ ([6]);
4) $R_s(G)\le\max\{3\cdot\mathsf{td}(G)-1,\ 2\cdot\mathsf{td}(G)+1\}$ for every $s\in V$ ([6]);
5) $\mathsf{tl}(G)-1\le\Delta_s(G)\le 3\cdot\mathsf{tl}(G)$, $R_s(G)\le 2\cdot\mathsf{tl}(G)$ for every $s\in V$ ([7,8]);
6) $\mathsf{tb}(G)-1\le R_s(G)\le 3\cdot\mathsf{tb}(G)$ ([10]);
7) $\mathsf{tl}(G)\le\mathsf{td}(G)\le\mathsf{ts}(G)$, $\mathsf{tb}(G)\le\lceil\mathsf{ts}(G)/2\rceil$ ([10]);
8) $\mathsf{ts}(G)\le 2\cdot\mathsf{tb}(G)\cdot\log_2 n$ and $\mathsf{ts}(G)\le 2\cdot\mathsf{td}(G)\cdot\log_2 n$ ([10]).

Inequalities in 2) and 3) imply that the tree-distortion td(G) of a δ-hyperbolic graph G is at most $O(\delta\log n)$. However, a stronger additive version of this result holds [4,5]: Every n-vertex δ-hyperbolic graph $G=(V,E)$ admits an unweighted tree $T=(V,U)$ (without Steiner points), constructible in linear time, such that $d_T(x,y)-2\le d_G(x,y)\le d_T(x,y)+O(\delta\log n)$ for any $x,y\in V$. Furthermore, it is easy to show that any graph G admitting a tree T with $d_G(x,y)\le d_T(x,y)\le d_G(x,y)+r$ for any $x,y\in V$ is r-hyperbolic. So, the hyperbolicity of a graph G is in fact an indicator of an embedabily of G in a tree with an additive distortion. It follows also from the inequalities listed that the tree-stretch ts(G) of a δ-hyperbolic graph G is at most $O(\delta\log^2 n)$.

While hb(G), $R_s(G)$, $\Delta_s(G)$ for a given graph G can be computed in polynomial time (in at most $O(n^4)$ time for hb(G) and in at most $O(nm)$ time for $R_s(G)$ and $\Delta_s(G)$ (see [2,3])) for any n-vertex, m-edge graph G, checking whether ts(G) is at most t and whether tl(G) is at most λ are NP-complete problems in general unweighted graphs for every $t > 3$ [1] and every $\lambda > 1$ [12] (similar NP-completeness results hold also for td(G) and tb(G)). The inequalities listed show that $\Delta_s(G)$ gives a near 3-approximation of tl(G) and of td(G) and an $O(\log n)$-approximation of ts(G), while $R_s(G)$ gives a near 3-approximation of tb(G).

The above inequalities and results provide not only efficiently computable bounds on those parameters but also serve as basis for constructing best approximation algorithms for the corresponding optimization problems which are NP-hard in general. For example, using the relationship between tl(G) and $\Delta_s(G)$ and the fact that a layering partition of a graph G can be constructed in linear time (see [3]), in [8] a linear time algorithm is provided which construct for a given graph G a Robertson-Seymour's tree-decomposition with bags of diameter at most $3 \cdot \mathsf{tl}(G) + 1$. Using the relationship between td(G) and $\Delta_s(G)$, in [6] an efficient 6-approximation algorithm was provided for the problem of minimum distortion embedding of a graph to a tree. The previous approximation bound was 27. Using the relationship between tb(G) and ts(G), in [10] an efficient $(\log_2 n)$-approximation algorithm was provided for the problem of constructing for a given graph G a tree t-spanner with minimum stretch t. Using the relationship between tb(G) and ts(G), [9] discusses also how to "turn", with a slight increase in the number of trees and in the stretch, a multiplicative tree spanner into a small set of collective additive tree spanners (see [11] for the definition).

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References

1. Cai, L., Corneil, D.G.: Tree spanners. SIAM J. Discrete Math. 8, 359–387 (1995)
2. Brandstädt, A., Chepoi, V.D., Dragan, F.F.: Distance approximating trees for chordal and dually chordal graphs. Journal of Algorithms 30, 166–184 (1999)
3. Chepoi, V.D., Dragan, F.F.: A note on distance approximating trees in graphs. European Journal of Combinatorics 21, 761–766 (2000)
4. Chepoi, V.D., Dragan, F.F., Estellon, B., Habib, M., Vaxes, Y.: Diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs. In: Proceedings of the 24th Annual ACM Symposium on Computational Geometry (SoCG 2008), College Park, Maryland, USA, June 9-11, pp. 59–68 (2008)
5. Chepoi, V.D., Dragan, F.F., Estellon, B., Habib, M., Vaxes, Y., Xiang, Y.: Additive Spanners and Distance and Routing Labeling Schemes for δ-Hyperbolic Graphs. Algorithmica 62(3-4), 713–732 (2012)
6. Chepoi, V.D., Dragan, F.F., Newman, I., Rabinovich, Y., Vaxes, Y.: Constant Approximation Algorithms for Embedding Graph Metrics into Trees and Outerplanar Graphs. Discr. & Comput. Geom. 47, 187–214 (2012)
7. Dourisboure, Y., Dragan, F.F., Gavoille, C., Yan, C.: Spanners for bounded tree-length graphs. Theoretical. Computer Science 383, 34–44 (2007)

8. Dourisboure, Y., Gavoille, C.: Tree-decompositions with bags of small diameter. Discr. Math. 307, 2008–2029 (2007)
9. Dragan, F.F., Abu-Ata, M.: Collective Additive Tree Spanners of Bounded Tree-Breadth Graphs with Generalizations and Consequences. In: van Emde Boas, P., Groen, F.C.A., Italiano, G.F., Nawrocki, J., Sack, H. (eds.) SOFSEM 2013. LNCS, vol. 7741, pp. 194–206. Springer, Heidelberg (2013)
10. Dragan, F.F., Köhler, E.: An Approximation Algorithm for the Tree t-Spanner Problem on Unweighted Graphs via Generalized Chordal Graphs. Algorithmica (in print), doi:10.1007/s00453-013-9765-4
11. Dragan, F.F., Yan, C., Lomonosov, I.: Collective tree spanners of graphs. SIAM J. Discrete Math. 20, 241–260 (2006)
12. Lokshtanov, D.: On the complexity of computing tree-length. Discrete Appl. Math. 158, 820–827 (2010)