Minimum Eccentricity Shortest Paths in Some Structured Graph Classes

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Abstract. We investigate the *Minimum Eccentricity Shortest Path* problem in some structured graph classes. It asks for a given graph to find a shortest path with minimum eccentricity. Although it is NP-hard in general graphs, we demonstrate that a minimum eccentricity shortest path can be found in linear time for distance-hereditary graphs (generalizing the previous result for trees) and in $\mathcal{O}(n^3m)$ time for chordal graphs.

1 Introduction

The *Minimum Eccentricity Shortest Path* problem asks for a given graph $G = (V, E)$ to find a shortest path P such that for each other shortest path Q, $\text{ecc}_G(P) \leq \text{ecc}_G(Q)$ holds. Here, the eccentricity of a set $S \subseteq V$ in G is $\text{ecc}_G(S) = \max_{u \in V} d_G(u, S)$. This problem was introduced in [7]. It may arise in determining a "most accessible" speedy linear route in a network and can find applications in communication networks, transportation planning, water resource management and fluid transportation. It was also shown in [6,7] that a minimum eccentricity shortest path plays a crucial role in obtaining the best to date approximation algorithm for a minimum distortion embedding of a graph into the line. Specifically, every graph G with a shortest path of eccentricity r admits an embedding f of G into the line with distortion at most $(8r + 2)\,\text{ld}(G)$, where $\text{ld}(G)$ is the minimum line-distortion of G (see [7] for details). Furthermore, if a shortest path of G of eccentricity r is given in advance, then such an embedding f can be found in linear time.

Those applications motivate investigation of the Minimum Eccentricity Shortest Path problem in general graphs and in particular graph classes. Fast algorithms for it will imply fast approximation algorithms for the minimum line distortion problem. Existence of low eccentricity shortest paths in structured graph classes will imply low approximation bounds for those classes. For example, all AT-free graphs (hence, all interval, permutation, cocomparability graphs) enjoy a shortest path of eccentricity at most 1 [4], all convex bipartite graphs enjoy a shortest path of eccentricity at most 2 [6].

In [7], the Minimum Eccentricity Shortest Path problem was investigated in general graphs. It was shown that its decision version is NP-complete (even for graphs with vertex degree at most 3). However, there are efficient approximation

E.W. Mayr (Ed.): WG 2015, LNCS 9224, pp. 189–202, 2016.
DOI: 10.1007/978-3-662-53174-7_14

algorithms: a 2-approximation, a 3-approximation, and an 8-approximation for the problem can be computed in $\mathcal{O}(n^3)$ time, in $\mathcal{O}(nm)$ time, and in linear time, respectively. Furthermore, a shortest path of minimum eccentricity r in general graphs can be computed in $\mathcal{O}(n^{2r+2}m)$ time. Paper [7] initiated also the study of the Minimum Eccentricity Shortest Path problem in special graph classes by showing that a minimum eccentricity shortest path in trees can be found in linear time. In fact, every diametral path of a tree is a minimum eccentricity shortest path.

In this paper, we design efficient algorithms for the Minimum Eccentricity Shortest Path problem in distance-hereditary graphs and in chordal graphs. We show that the problem can be solved in linear time for distance-hereditary graphs (generalizing the previous result for trees) and in $\mathcal{O}(n^3m)$ time for chordal graphs.

Note that our Minimum Eccentricity Shortest Path problem is close but different from the *Central Path* problem in graphs introduced in [13]. It asks for a given graph G to find a path P (not necessarily shortest) such that any other path of G has eccentricity at least $\mathrm{ecc}_G(P)$. The Central Path problem generalizes the Hamiltonian Path problem and therefore is NP-hard even for chordal graphs [12]. Our problem is polynomial time solvable for chordal graphs.

2 Notions and Notations

All graphs occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. For a graph $G = (V, E)$, we use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and the edge set of G. $G[S]$ denotes the *induced subgraph* of G with the vertex set S.

The *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ of two vertices u and v is the length of a shortest path connecting u and v. The distance between a vertex v and a set $S \subseteq V$ is defined as $d_G(v, S) = \min_{u \in S} d_G(u, v)$. The *eccentricity* $\mathrm{ecc}_G(v)$ of a vertex v is $\max_{u \in V} d_G(u, v)$. For a set $S \subseteq V$, its eccentricity is $\mathrm{ecc}_G(S) = \max_{u \in V} d_G(u, S)$. If no ambiguity arises, we will omit the subscript G. For a vertex pair s, t, a shortest (s, t)-path P has *minimal eccentricity*, if there is no shortest (s, t)-path Q with $\mathrm{ecc}(Q) < \mathrm{ecc}(P)$. Two vertices x and y are called *mutually furthest* if $d_G(x, y) = \mathrm{ecc}(x) = \mathrm{ecc}(y)$. A vertex u is *k-dominated* by a vertex v (by a set $S \subset V$), if $d_G(u, v) \leq k$ ($d_G(u, S) \leq k$, respectively).

The *diameter* of a graph G is $\mathrm{diam}(G) = \max_{u,v \in V} d_G(u, v)$. The diameter $\mathrm{diam}_G(S)$ of a set $S \subseteq V$ is defined as $\max_{u,v \in S} d_G(u, v)$. A pair of vertices x, y of G is called a *diametral pair* if $d_G(x, y) = \mathrm{diam}(G)$. In this case, every shortest path connecting x and y is called a *diametral path*.

For a vertex $v \in V$, $N(v) = \{u \in V \mid uv \in E\}$ is called the *open neighborhood*, and $N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. $N^r[v] = \{u \in V \mid d_G(u, v) \leq r\}$ denotes the *disk* of radius r around vertex v. Additionally, $L_r^{(v)} = \{u \in V \mid d_G(u, v) = i\}$ denotes the vertices with distance r from v. For two vertices u and v, $I(u, v) = \{w \mid d_G(u, v) = d_G(u, w) + d_G(w, v)\}$

is the *interval* between u and v. The set $S_i(s,t) = L_i^{(s)} \cap I(u,v)$ is called a *slice* of the interval from u to v. For any set $S \subseteq V$ and a vertex v, $\Pr(v,S) = \{u \in S \mid d_G(u,v) = d_G(v,S)\}$ denotes the *projection* of v on S.

A *chord* in a path is an edge connecting two non-consecutive vertices of the path. A set of vertices S is a *clique* if all vertices in S are pairwise adjacent. A graph is *chordal* if every cycle with at least four vertices has a chord. A graph is *distance-hereditary* if the distances in any connected induced subgraph are the same as they are in the original graph. For more definitions of these classes and relations between them see [2].

3 A Linear-Time Algorithm for Distance-Hereditary Graphs

Distance-hereditary graphs can be defined as graphs where each chordless path is a shortest path [10]. Several interesting characterizations of distance-hereditary graphs in terms of metric and neighborhood properties, and forbidden configurations were provided by Bandelt and Mulder [1], and by D'Atri and Moscarini [5]. The following proposition lists the basic information on distance-hereditary graphs that is needed in what follows.

Proposition 1 ([1,5]). *For a graph G the following conditions are equivalent:*

(1) G is distance-hereditary;
(2) The house, domino, gem (see Fig. 1) and the cycles C_k of length $k \geq 5$ are not induced subgraphs of G;
(3) For an arbitrary vertex x of G and every pair of vertices $u, v \in L_k^{(x)}$, that are in the same connected component of the graph $G[V \setminus L_{k-1}^{(x)}]$, we have $N(v) \cap L_{k-1}^{(x)} = N(u) \cap L_{k-1}^{(x)}$.
(4) (4-point condition) For any four vertices u, v, w, x of G at least two of the following distance sums are equal: $d_G(u,v) + d_G(w,x)$; $d_G(u,w) + d_G(v,x)$; $d_G(u,x) + d_G(v,w)$. If the smaller sums are equal, then the largest one exceeds the smaller ones at most by 2.

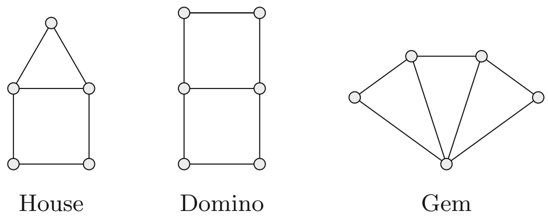

Fig. 1. Forbidden induced subgraphs in a distance-hereditary graph.

As a consequence of statement (3) of Proposition 1 we get.

Corollary 1. *Let $P := P(s,t)$ be a shortest path in a distance-hereditary graph G connecting vertices s and t, and w be an arbitrary vertex of G. Let a be a vertex of $\Pr(w,P)$ that is closest to s, and let b be a vertex of $\Pr(w,P)$ that is closest to t. Then $d_G(a,b) \leq 2$ and there must be a vertex w' in G adjacent to both a and b and at distance $d_G(w,P)-1$ from w.*

As a consequence of statement (4) of Proposition 1 we get.

Corollary 2. *Let x, y, v, u be arbitrary vertices of a distance-hereditary graph G with $v \in I(x,u)$, $u \in I(y,v)$, and $d_G(u,v) > 1$, then $d_G(x,y) = d_G(x,v) + d_G(v,u) + d_G(u,y)$. That is, if two shortest paths share ends of length at least 2, then their union is a shortest path.*

Proof. Consider distance sums $S_1 := d_G(x,v) + d_G(u,y)$, $S_2 := d_G(x,y) + d_G(u,v)$ and $S_3 := d_G(x,u) + d_G(v,y)$. Since $d_G(x,u) + d_G(v,y) = d_G(x,v) + d_G(u,y) + 2\,d_G(u,v)$, we have $S_3 > S_1$. Then, either $S_2 = S_3$ or $S_1 = S_2$ and $S_3 - S_1 \leq 2$. If the latter is true, then $2 \geq S_3 - S_1 = d_G(x,v) + d_G(u,y) + 2\,d_G(u,v) - d_G(x,v) - d_G(u,y) = 2\,d_G(v,u) > 2$ and a contradiction arises. Thus, $S_2 = S_3$ and we get $d_G(x,y) = d_G(x,v) + d_G(v,u) + d_G(u,y)$. □

Lemma 1. *Let x, y be a diametral pair of vertices of a distance-hereditary graph G, and k be the minimum eccentricity of a shortest path in G. If for some shortest path $P = P(x,y)$, connecting x and y, $\mathrm{ecc}(P) > k$ holds, then $\mathrm{diam}(G) = d_G(x,y) \geq 2k$. Furthermore, if $d_G(x,y) = 2k$ then there is a shortest path P^* between x and y with $\mathrm{ecc}(P^*) = k$.*

Proof. Consider a vertex v with $d_G(v,P) > k$. Let x' be a vertex of $\Pr(v,P)$ closest to x, and y' be a vertex of $\Pr(v,P)$ closest to y. By Corollary 1, $d_G(x',y') \leq 2$ and there must be a vertex v' in G adjacent to both x' and y' and at distance $d_G(v,P)-1$ from v. Let $P(x,x')$ and $P(y',y)$ be subpaths of P connecting vertices x, x' and vertices y, y', respectively. Consider also an arbitrary shortest path $Q(v,v')$ connecting v and v' in G. By choices of x' and y', no chords in G exist in paths $P(x,x') \cup Q(v',v)$ and $P(y,y') \cup Q(v',v)$. Hence, those paths are shortest in G. Since x, y is a diametral pair, we have $d_G(x,x') + d_G(x',y') + d_G(y',y) = d_G(x,y) \geq d_G(x,v) = d_G(x,x') + 1 + d_G(v',v)$. That is, $d_G(y',y) \geq d_G(v',v) + 1 - d_G(x',y')$. Similarly, $d_G(x',x) \geq d_G(v',v) + 1 - d_G(x',y')$. Combining both inequalities and taking into account that $d_G(v,v') \geq k$, we get $d_G(x,y) = d_G(x,x') + d_G(x',y') + d_G(y',y) \geq 2k + 2 - d_G(x',y') \geq 2k$. Furthermore, we have $d_G(x,y) \geq 2k+1$ if $d_G(x',y') = 1$ and $d_G(x,y) \geq 2k+2$ if $d_G(x',y') = 0$. Also, if $d_G(x,y) = 2k$ then $d_G(x',y') = 2$, $d_G(v,v') = k$, $d_G(x,x') = d_G(y,y') = k-1$ and $d_G(v,x) = d_G(v,y) = 2k$.

Now assume that $d_G(x,y) = 2k$. Consider sets $S = \{w \in V \mid d_G(x,w) = d_G(y,w) = k\}$ and $F_{x,y} = \{u \in V \mid d_G(u,x) = d_G(u,y) = 2k\}$. Let $c \in S$ be a vertex of S that k-dominates the maximum number of vertices in $F_{x,y}$. Consider a shortest path P^* connecting vertices x and y and passing through vertex c.

We will show that $\operatorname{ecc}(P^*) = k$. Let x' (y') be the neighbor of c in subpath of P^* connecting c with x (with y, respectively).

Assume there is a vertex v in G such that $d_G(v, P^*) > k$. As in the first part of the proof, one can show that $d_G(v, x') = d_G(v, y') = k + 1$, i.e., $x', y' \in \Pr(v, P^*)$ and $d_G(v, P^*) = k + 1$. Furthermore, $d_G(v, x) = d_G(v, y) = 2k$, i.e., $v \in F_{x,y}$. Also, vertex v', that is adjacent to x', y' and at distance k from v, must belong to S. Since $d_G(v, c) > k$ but $d_G(v, v') = k$, by choice of c, there must exist a vertex $u \in F_{x,y}$ such that $d_G(u, c) \le k$ and $d_G(u, v') > k$. Since $d_G(u, y) = d_G(u, x) = 2k$, $d_G(u, c)$ must equal k and both $d_G(u, x')$ and $d_G(u, y')$ must equal $k + 1$.

Since $d_G(v, u) \le \operatorname{diam}(G) = 2k$ and $d_G(v, y') = d_G(v, x') = k + 1 = d_G(u, x') = d_G(u, y')$, we must have a chord between vertices of a shortest path $P(v, v')$ connecting v with v' and vertices of a shortest path $P(u, c)$ connecting u with c. If no chords exist or only chord cv' is present, then $d_G(v, u) \ge 2k + 1$, contradicting with $\operatorname{diam}(G) = 2k$. So, consider a chord ab with $a \in P(v, v')$, $b \in P(u, c)$, $ab \ne cv'$, and $d_G(a, v') + d_G(b, c)$ is minimum. We know that $d_G(a, v') = d_G(b, c)$ must hold since $d_G(u, v') > k = d_G(u, c)$ and $d_G(v, c) > k = d_G(v, v')$. To avoid induced cycles of length $k \ge 5$, $d_G(a, v') = d_G(b, c) = 1$ must hold. But then, vertices a, b, c, x', v' form either an induced cycle C_5, when c and v' are not adjacent, or a house, otherwise. Note that, by distance requirements, edges bv', ca, bx', and ax' are not possible.

Contradictions obtained show that such a vertex v with $d_G(v, P^*) > k$ is not possible, i.e., $\operatorname{ecc}(P^*) = k$. □

Lemma 2. *In every distance-hereditary graph there is a minimum eccentricity shortest path $P(s, t)$ where s and t are two mutually furthest vertices.*

Proof. Let k be the minimum eccentricity of a shortest path in G. Let $Q := Q(s, t) = (s = v_0, v_1, \ldots, v_i, \ldots, v_q = t)$ be a shortest path of G of eccentricity k with maximum q, that is, among all shortest paths with eccentricity k, Q is a longest one. Assume, without loss of generality, that t is not a vertex most distant from s. Let $i \le q$ be the smallest index such that subpath $Q(s, v_i) = (v_0, v_1, \ldots, v_i)$ of Q has also the eccentricity k. By choice of i, there must exist a vertex v in G which is k-dominated only by vertex v_i of $Q(s, v_i)$, i.e., $\Pr(v, Q(s, v_i)) = \{v_i\}$ and $d_G(v, Q(s, v_i)) = k$. Let $P(v, v_i)$ be an arbitrary shortest path of G connecting v with v_i. By choice of i, no vertex of $P(v, v_i) \setminus \{v_i\}$ is adjacent to a vertex of $Q(s, v_i) \setminus \{v_i\}$. Hence, path obtained by concatenating $Q(s, v_i)$ with $P(v_i, v)$ is chordless and, therefore, shortest in G, and has eccentricity k, too. Note that v is now a most distant vertex from s, i.e., $d_G(s, v) = \operatorname{ecc}(s)$. Since $d_G(s, v) > d_G(s, t)$, a contradiction with maximality of q arises. □

The main result of this section is the following.

Theorem 1. *Let x, y be a diametral pair of vertices of a distance-hereditary graph G, and k be the minimum eccentricity of a shortest path in G. Then, there is a shortest path P between x and y with* $\operatorname{ecc}(P) = k$.

Proof. We may assume that for some shortest path P' connecting x and y, $\text{ecc}(P') > k$ holds (otherwise, there is nothing to prove). Then, by Lemma 1, we have $d(x, y) \geq 2k$.

Let $Q := Q(s, t) = (s = v_0, v_1, \ldots, v_i, \ldots, v_q = t)$ be a shortest path of G of eccentricity k such that s and t are two mutually furthest vertices (see Lemma 2). Consider projections of x and y to Q. We distinguish between three cases: $\Pr(x, Q)$ is completely on the left of $\Pr(y, Q)$ in Q; $\Pr(x, Q)$ and $\Pr(y, Q)$ have a common vertex w; and the remaining case (see Corollary 1) when $\Pr(x, Q) = \{v_{i-1}, v_{i+1}\}$ and $\Pr(y, Q) = \{v_i\}$ for some index i.

Case 1: $\Pr(x, Q)$ *is completely on the left of* $\Pr(y, Q)$ *in* Q.
Let x' be a vertex of $\Pr(x, Q)$ closest to t and y' a vertex of $\Pr(y, Q)$ closest to s. Consider an arbitrary shortest path $P(x, x')$ of G connecting vertices x and x', an arbitrary shortest path $P(y', y)$ of G connecting vertices y' and y, and a subpath $Q(x', y')$ of $Q(s, t)$ between vertices x' and y'. We claim that the path P of G obtained by concatenating $P(x, x')$ with $Q(x', y')$ and then with $P(y', y)$ is a shortest path of eccentricity k.

Indeed, by choice of x', no edge connecting a vertex in $P(x, x') \setminus \{x'\}$ with a vertex in $Q(x', y') \setminus \{x'\}$ can exist in G. Similarly, no edge connecting a vertex in $P(y', y) \setminus \{y'\}$ with a vertex in $Q(x', y') \setminus \{y'\}$ can exist in G. Since we also have $d_G(x, y) \geq 2k$, $d_G(x, Q) \leq k$ and $d_G(y, Q) \leq k$, no edge connecting a vertex in $P(y', y) \setminus \{y'\}$ with a vertex in $P(x, x') \setminus \{x'\}$ can exist in G. Hence, chordless path $P = P(x, x') \cup Q(x', y') \cup P(y', y)$ is a shortest path of G.

Consider now an arbitrary vertex v of G. We want to show that $d_G(v, P) \leq k$. Since $\text{ecc}(Q) = k$, $d_G(v, Q) \leq k$. Consider the projection of v to Q. We may assume that $\Pr(v, Q) \cap Q(x', y') = \emptyset$ and, without loss of generality, that vertices of $\Pr(v, Q)$ are closer to s than vertex x'. Let v' be a vertex of $\Pr(v, Q)$ closest to x'. As before, by choices of v' and y', paths $P(y, y') \cup Q(y', v')$ and $P(v, v') \cup Q(y', v')$ are chordless and, therefore, are shortest paths of G (here $P(v, v')$ is an arbitrary shortest path of G connecting v with v'). Since $d_G(v', y') \geq 2$, by Corollary 2, $d_G(v, y) = d_G(v, v') + d_G(v', y') + d_G(y', y)$. Hence, from $d_G(x, y) \geq d_G(y, v)$, $d_G(x, y) = d_G(x, x') + d_G(x', y)$ and $d_G(v, y) = d_G(v, x') + d_G(x', y)$, we obtain $d_G(v, x') \leq d_G(x, x') \leq k$.

Case 2: $\Pr(x, Q)$ *and* $\Pr(y, Q)$ *have a common vertex* w.

In this case, we have $d_G(x, y) \leq d_G(x, w) + d_G(y, w) \leq k + k = 2k$. Earlier we assumed also that $d_G(x, y) \geq 2k$. Hence, $\text{diam}(G) = d_G(x, y) = 2k$ and the statement of the theorem follows from Lemma 1.

Case 3: Remaining case when $\Pr(x, Q) = \{v_{i-1}, v_{i+1}\}$ *and* $\Pr(y, Q) = \{v_i\}$ *for some index* i.

In this case, we have $d_G(x, y) \leq d_G(x, v_{i-1}) + 1 + d_G(v_i, y) \leq 2k + 1$. By Lemma 1, we can assume that $\text{diam}(G) = d_G(x, y) = 2k + 1$, i. e., $d_G(x, v_{i-1}) = d_G(x, v_{i+1}) = d_G(v_i, y) = k$.

Let $Q(s, v_{i-1})$ and $Q(t, v_{i+1})$ be subpaths of Q connecting vertices s and v_{i-1} and vertices t and v_{i+1}, respectively. Pick an arbitrary shortest path $P(y, v_i)$ connecting y with v_i. Since no chords are possible between $Q(s, v_i) \setminus \{v_i\}$ and

$P(y, v_i) \setminus \{v_i\}$ and between $Q(t, v_i) \setminus \{v_i\}$ and $P(y, v_i) \setminus \{v_i\}$, we have $d_G(y, t) = d_G(y, v_i) + d_G(v_i, t) = k + d_G(v_i, t)$ and $d_G(y, s) = d_G(y, v_i) + d_G(v_i, s) = k + d_G(v_i, s)$. Inequalities $d_G(x, y) \geq d_G(y, t)$ and $d_G(x, y) \geq d_G(y, s)$ imply $d_G(v_{i+1}, t) \leq d_G(v_{i+1}, x) = k$ and $d_G(v_{i-1}, s) \leq d_G(v_{i-1}, x) = k$. If both $d_G(v_{i+1}, t)$ and $d_G(v_{i-1}, s)$ equal k, then $d_G(s, t) = 2k + 2$ contradicting with $\mathrm{diam}(G) = 2k + 1$. Hence, we may assume, without loss of generality, that $d_G(v_{i-1}, s) \leq k - 1$. We will show that shortest path $P := P(x, v_{i+1}) \cup P(v_i, y)$ has eccentricity k (here, $P(x, v_{i+1})$ is an arbitrary shortest path of G connecting x with v_{i+1}).

Consider a vertex v in G and assume that $\mathrm{Pr}(v, Q)$ is strictly contained in $Q(t, v_{i+1})$. Denote by v' the vertex of $\mathrm{Pr}(v, Q)$ that is closest to s. Let $P(v, v')$ be an arbitrary shortest path connecting v and v'. As before, $P(v, v') \cup Q(v', s)$ is a chordless path and therefore $d_G(v, s) = d_G(v, v_{i+1}) + d_G(v_{i+1}, s)$. Since t is a most distant vertex from s, $d_G(s, v) \leq d_G(s, t)$. Hence, $d_G(v, v_{i+1}) + d_G(v_{i+1}, s) = d_G(s, v) \leq d_G(s, t) = d_G(s, v_{i+1}) + d_G(v_{i+1}, t)$, i.e., $d_G(v, v_{i+1}) \leq d_G(v_{i+1}, t) \leq k$.

Consider a vertex v in G and assume now that $\mathrm{Pr}(v, Q)$ is strictly contained in $Q(s, v_{i-1})$. Denote by v' the vertex of $\mathrm{Pr}(v, Q)$ that is closest to t. Let $P(v, v')$ be an arbitrary shortest path connecting v and v'. Again, $P(v, v') \cup Q(v', t)$ is a chordless path and therefore $d_G(v, t) = d_G(v, v_i) + d_G(v_i, t)$. Since s is a most distant vertex from t, $d_G(t, v) \leq d_G(s, t)$. Hence, $d_G(v, v_i) + d_G(v_i, t) = d_G(t, v) \leq d_G(s, t) = d_G(s, v_i) + d_G(v_i, t)$, i.e., $d_G(v, v_i) \leq d_G(v_i, s) \leq k$.

Thus, all vertices of G are k-dominated by $P(x, v_{i+1}) \cup P(v_i, y)$. □

It is known [8] that a diametral pair of a distance-hereditary graph can be found in linear time. Hence, according to Theorem 1, to find a shortest path of minimum eccentricity in a distance-hereditary graph in linear time, one needs to efficiently extract a best eccentricity shortest path for a given pair of end-vertices. In what follows, we demonstrate that, for a distance-hereditary graph, such an extraction can be done in linear time as well.

We will need few auxiliary lemmas.

Lemma 3. *In a distance-hereditary graph G, for each pair of vertices s and t, if x is on a shortest path from v to $\Pi_v = \mathrm{Pr}(v, I(s, t))$ and $d_G(x, \Pi_v) = 1$, then $\Pi_v \subseteq N(x)$.*

Proof. Let p and q be two vertices in Π_v and $d_G(v, \Pi_v) = r$. By statement (3) of Proposition 1, $N(p) \cap L_{r-1}^{(v)} = N(q) \cap L_{r-1}^{(v)}$. Thus, each vertex x on a shortest path from v to Π_v with $d_G(x, \Pi_v) = 1$ (which is in $N(p) \cap L_{r-1}^{(v)}$ by definition) is adjacent to all vertices in Π_v, i.e., $\Pi_v \subseteq N(x)$. □

Lemma 4. *In a distance-hereditary graph G, let $S_i(s, t)$ and $S_{i+1}(s, t)$ be two consecutive slices of an interval $I(s, t)$. Each vertex in $S_i(s, t)$ is adjacent to each vertex in $S_{i+1}(s, t)$.*

Proof. Consider statement (3) of Proposition 1 from perspective of t. Thus, $S_i(s, t) \subseteq N(v)$ for each vertex $v \in S_{i+1}(s, t)$. Additionally, from perspective of s, $S_{i+1}(s, t) \subseteq N(u)$ for each vertex $u \in S_i(s, t)$. □

Lemma 5. *In a distance-hereditary graph G, if a projection $\Pi_v = \Pr(v, I(s,t))$ intersects two slices of an interval $I(s,t)$, each shortest (s,t)-path intersects Π_v.*

Proof. Because of Lemma 3, there is a vertex x with $N(x) \supseteq \Pi_v$ and $d_G(v,x) = d_G(v,\Pi_v) - 1$. Thus, Π_v intersects at most two slices of interval $I(s,t)$ and those slices have to be consecutive, otherwise x would be a part of the interval. Let $S_i(s,t)$ and $S_{i+1}(s,t)$ be these slices. Note that $d_G(s,x) = i+1$. Thus, by statement (3) of Proposition 1, $N(x) \cap S_i(s,t) = N(u) \cap S_i(s,t)$ for each $u \in S_{i+1}(s,t)$. Therefore, $S_i(s,t) \subseteq \Pi_v$, i.e., each shortest path from s to t intersects Π_v. □

From the lemmas above, we can conclude that, for determining a shortest (s,t)-path with minimal eccentricity, a vertex v is only relevant if $d_G(v, I(s,t)) = \mathrm{ecc}(I(s,t))$ and the projection of v on the interval $I(s,t)$ only intersects one slice. Algorithm 1 uses this.

Algorithm 1. Computes a shortest (s,t)-path P with minimal eccentricity for a given distance-hereditary graph G and a vertex pair s,t.

Input: A distance-hereditary graph $G = (V,E)$ and two distinct vertices s and t.

Output: A shortest path P from s to t with minimal eccentricity.

1 Compute the sets $V_i = \{v \mid d_G(v, I(s,t)) = i\}$ for $1 \le i \le \mathrm{ecc}(I(s,t))$.
2 Each vertex $v \notin I(s,t)$ gets a pointer $g(v)$ initialised with $g(v) := v$ if $v \in V_1$, and $g(v) := \varnothing$ otherwise.
3 **for** $i := 2$ **to** $\mathrm{ecc}(I(s,t))$ **do**
4   For each $v \in V_i$, select a vertex $u \in V_{i-1} \cap N(v)$ and set $g(v) := g(u)$.
5 **foreach** $v \in V_{\mathrm{ecc}(I(s,t))}$ **do**
6   If $N(g(v))$ intersects only one slice of $I(s,t)$, flag $g(v)$ as *relevant*.
7 Set $P := \{s,t\}$.
8 **for** $i := 1$ **to** $d_G(s,t) - 1$ **do**
9   Find a vertex $v \in S_i(s,t)$ for which the number of *relevant* vertices in $N(v)$ is maximal.
10   Add v to P.

Lemma 6. *For a distance-hereditary graph G and an arbitrary vertex pair s,t, Algorithm 1 computes a shortest (s,t)-path with minimal eccentricity in linear time.*

Proof. The loop in line 3 determines for each vertex v outside of the interval $I(s,t)$ a *gate vertex* $g(v)$ such that $N(g(v)) \supseteq \Pr(v, I(s,t))$ and $d_G(v, I(s,t)) = d_G(v, g(v)) + 1$ (see Lemma 3). From Lemmas 5 and 4, it follows that for a vertex v which is not in $V_{\mathrm{ecc}(I(s,t))}$ or its projection to $I(s,t)$ is intersecting two slices of $I(s,t)$, $d_G(v, P(s,t)) \le \mathrm{ecc}(I(s,t))$ for every shortest path $P(s,t)$ between s and t. Therefore, line 6 only marks $g(v)$ if $v \in V_{\mathrm{ecc}(I(s,t))}$ and its projection $\Pr(v, I(s,t))$ intersects only one slice. Because only one slice is intersected and each vertex in a slice is adjacent to all vertices in the consecutive slice (see Lemma 4), in each slice the vertex of an optimal (of minimum eccentricity) path P can be selected independently from the preceding vertex. If a vertex x of a slice $S_i(s,t)$ has the maximum number of *relevant* vertices in $N(x)$, then x is good to put in P. Indeed, if x dominates all relevant vertices adjacent to vertices of $S_i(s,t)$, then x is a perfect choice to put in P. Else, any vertex y of a slice $S_i(s,t)$ is a good vertex to put in P. Hence, P is optimal if the number of *relevant* vertices adjacent to P is maximal. Thus, the path selected in line 8 to line 10 is optimal. □

Running Algorithm 1 for a diametral pair of vertices of a distance-hereditary graph G, by Theorem 1, we get a shortest path of G with minimum eccentricity. Thus, we have proven the following result.

Theorem 2. *A shortest path with minimum eccentricity of a distance-hereditary graph $G = (V, E)$ can be computed in $\mathcal{O}(|V| + |E|)$ total time.*

4 A Polynomial-Time Algorithm for Chordal Graphs

In what follows, we will show that the minimum eccentricity shortest path problem for chordal graphs can be solved in polynomial time.

For distance-hereditary graphs, we were able to show that there is a shortest path with minimum eccentricity between a diametral pair of vertices. This is not always the case for chordal graphs. Consider the graph G given in Fig. 2. The only diametral path in G is from s to w. Because of u, it has eccentricity 3. However, a shortest path from s to v containing t has eccentricity 2 which is optimal for G.

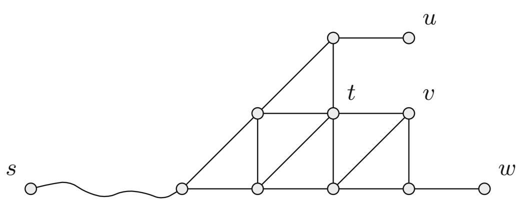

Fig. 2. A chordal graph for which no diametral path has the optimal eccentricity.

To find an optimal path, we create a simpler graph H for given start and end vertices s and t of a chordal graph G. Then, each shortest path P from s to t in H has eccentricity at most 2. Additionally, if P has minimal eccentricity in H, the corresponding path in G also has minimal eccentricity. Repeating this for each vertex pair s, t in G, we can find the minimum eccentricity shortest path of G.

The following lemmas allow us to create H.

Lemma 7 **([3]).** *For every chordal graph G and any two of its vertices s and t, each slice $S_i(s,t)$ is a clique.*

Corollary 3. *For each shortest path P from s to t in a chordal graph G,*

$$\mathrm{ecc}(I(s,t)) \leq \mathrm{ecc}(P) \leq \mathrm{ecc}(I(s,t)) + 1.$$

Lemma 8 **([9]).** *Let G be a chordal graph. If for two distinct vertices x, y in a disk $N_G^r[v]$ there is a path P connecting them with $P \cap N_G^r[v] = \{x, y\}$, then x and y are adjacent.*

Lemma 9. *Let G be a chordal graph. For each vertex $v \notin I(s,t)$, if a projection $\Pi_v = \mathrm{Pr}(v, I(s,t))$ is not a clique, then each shortest path from s to t intersects Π_v.*

Proof. Because Π_v is not a clique, there are two distinct vertices $u_i \in S_i(s,t) \cap \Pi_v$ and $u_j \in S_j(s,t) \cap \Pi_v$ which are not adjacent to each other. Consider an arbitrary shortest path Q from s to t and two vertices $q_i \in S_i(s,t) \cap Q$ and $q_j \in S_j(s,t) \cap Q$. Because each slice is a clique (see Lemma 7), there is a path $Q' = \{u_i\} \cup Q(q_i, q_j) \cup \{u_j\}$ from u_i to u_j. Note that Π_v is the intersection of $I(s,t)$ with the disk $N_G^r[v]$ (for $r = d_G(v, I(s,t))$). Thus, if Q and Π_v do not intersect, then $Q' \cap N_G^r[v] = \{u_i, u_j\}$. However, because u_i and u_j are not adjacent, this contradicts with Lemma 8. Therefore, Q and Π_v intersect. □

The conclusion from Corollary 3 and Lemma 9 is that a vertex v is only relevant for determining a minimal eccentricity shortest path from s to t, if $d_G(v, I(s,t)) = \mathrm{ecc}(I(s,t))$ and the projection of v on $I(s,t)$ intersects at most two slices. Therefore, we can create a graph H for a given chordal graph G using Algorithm 2. We call H a *hedgehog graph* for G.

Algorithm 2. Creates a hedgehog graph H from a given chordal graph G for its vertex pair s, t.

Input: A chordal graph $G = (V_G, E_G)$, and a vertex pair s, t.
Output: A hedgehog graph $H = (V_H, E_H)$.

Initialise $V_H := \emptyset$ and $E_H := \emptyset$.

Add $I_G(s,t)$ to H, i.e. $V_H := V_H \cup I_G(s,t)$ and $E_H := E_H \cup \{uv \in E_G \mid u, v \in I_G(s,t)\}$.

foreach $v \in V_G$ *with* $d_G(v, I_G(s,t)) = \mathrm{ecc}_G(I_G(s,t))$ **do**

  If $\mathrm{Pr}_G(v, I_G(s,t))$ intersects at most two slices of $I_G(s,t)$, then create a new vertex $g(v)$, add to H, and connect it with every vertex in $\mathrm{Pr}_G(v, I_G(s,t))$, i.e. $V_H := V_H \cup \{g(v)\}$ and $E_H := E_H \cup \{ug(v) \mid u \in \mathrm{Pr}_G(v, I_G(s,t))\}$.

Theorem 3. *For a chordal graph G and a vertex pair s, t, let H be the hedgehog graph of G created by Algorithm 2. A shortest (s,t)-path P in H has eccentricity* 1 *if and only if P has eccentricity* $\mathrm{ecc}_G(I_G(s,t))$ *in G.*

Proof. Assume $\mathrm{ecc}_H(P) = 1$. Therefore, for all vertices $g(v) \in V_H \setminus I_H(s,t)$, P intersects the projection $\mathrm{Pr}_H(g(v), I_H(s,t))$. Based on the construction of H, $\mathrm{Pr}_H(g(v), I_H(s,t)) = \mathrm{Pr}_G(v, I_G(s,t))$. Thus, $d_G(v,P) = \mathrm{ecc}_G(I_G(s,t))$ for all $v \in V_G$ with $d_G(v, I_G(s,t)) = \mathrm{ecc}_G(I_G(s,t))$ and projection $\mathrm{Pr}_G(v, I_G(s,t))$ intersecting at most two slices of $I_G(s,t)$. For all other $v \in V_G$, $d_G(v,P) \leq \mathrm{ecc}_G(I_G(s,t))$ follows from Corollary 3 and Lemma 9. Thus, $\mathrm{ecc}_G(P) = \mathrm{ecc}_G(I_G(s,t))$.

Assume $\mathrm{ecc}_H(P) > 1$. Thus, there is a vertex $v \in V_H \setminus I_H(s,t)$ such that P does not intersect the projection $\mathrm{Pr}_H(g(v), I_H(s,t))$. Therefore, $d_G(v,P) > d_G(v, I_G(s,t)) = \mathrm{ecc}_G(I_G(s,t))$. □

For the analysis of the complexity of Algorithm 2, we assume that the distance between any two vertices can be determined in constant time (i. e., the distance matrix of the graph is given). Computing the interval $I_G(s,t)$ and $\mathrm{ecc}_G(I_G(s,t))$ can be done in $\mathcal{O}(m)$ total time. For a given vertex v, $d_G(v, I_G(s,t))$ (line 3) and $\mathrm{Pr}_G(v, I_G(s,t))$ (line 4) can be calculated in $\mathcal{O}(n)$ time by determining the distance to all vertices in $I_G(s,t)$. Repeating this for all vertices in V_G leads to a total runtime of $\mathcal{O}(n^2)$.

After generating H, we need to determine if there is a shortest path from s to t in H with eccentricity 1.

Algorithm 3. Finds a shortest (s,t)-path with minimal eccentricity in a hedgehog graph H of a chordal graph.

Input: A hedgehog graph $H = (V,E)$ of a chordal graph and a vertex pair s, t.
Output: A shortest path P from s to t with minimal eccentricity.

For each vertex v and each edge e in H, set $\omega(v) := 0$ and $\omega(e) := 0$.
foreach $u \notin I_H(s,t)$ **do**
    if $N_H(u) \subseteq S_i(s,t)$ *(for some i)* **then**
        Set $\omega(v) := \omega(v) + 1$ for all $v \in N_H(u)$.
    else
        foreach $vw \in E$ *with* $d_H(u, vw) = 1$ *and* $d_H(s,u) = d_H(s,v) + 1 = d_H(s,w)$ **do**
            $\omega(vw) := \omega(vw) + 1$
Find a shortest path P from s to t such that the sum of all vertex and edge weights of P is maximal.

Lemma 10. *For a given vertex pair s, t and the corresponding hedgehog graph H, Algorithm 3 determines a shortest (s,t)-path of H with minimal eccentricity (one or two) in $\mathcal{O}(nm)$ time.*

Proof (Correctness). Let P be an arbitrary shortest path from s to t in H. We say $\omega(P) = \sum_{u \in P} \omega(u) + \sum_{vw \in E, vw \in P} \omega(vw)$ is the total weight of P. Because H is based on a chordal graph, each slice of $I_H(s,t)$ is a clique. Thus, each path from s to t has eccentricity at most 2. To proof the correctness of the algorithm, we will show that the total weight of P is equal to the number of vertices adjacent to P which are not part of the interval, i. e. $\omega(P) = |N_H[P] \setminus I_H(s,t)|$.

The vertices of H can be partitioned into the following three sets: $I_H(s,t)$, V_1 containing all vertices x whose $N_H(x)$ intersects only one slice of $I_H(s,t)$, and V_2 containing all vertices x whose $N_H(x)$ intersects two slices of $I_H(s,t)$.

For each vertex $u \in V_1$ the weight of every of its neighbors v is increased by 1 (line 4). Thus, $\omega(v) = |N_H(v) \cap V_1|$. Note that for $v, v' \in I_H(s,t)$, $d_G(s,v) \neq d_G(s,v')$ implies $N_H(v) \cap N_H(v') \cap V_1 = \emptyset$. Therefore, $\sum_{v \in P} \omega(v)$ is the number of vertices in V_1 which are adjacent to P.

Let $u \in V_2$ be a vertex such that $N_H(u)$ intersects the slices $S_i(s,t)$ and $S_{i+1}(s,t)$. Then the weight of all edges vw from $S_i(s,t)$ to $S_{i+1}(s,t)$ which intersect $N_H(u)$ is increased by 1 (line 7). Because the weight of an edge vw is only increased if $d_H(s,u) = d_H(s,v) + 1 = d_H(s,w)$, $\omega(vw) = |N_H(vw) \cap V_2 \cap L^{(s)}_{i+1}|$. Therefore, $\sum_{vw \in E, vw \in P} \omega(vw)$ is the number of vertices in V_2 which are adjacent to P.

It follows that each vertex in $N_H[P] \setminus I_H(s,t)$ is counted exactly once for the total weight of P. Therefore, P has eccentricity 1 if and only if $\omega(P) = |V_1 \cup V_2|$. □

Proof (Complexity). Initialising the vertex and edge weights can be done in linear time. For a vertex $u \notin I_H(s,t)$, line 4 only updates the neighborhood of u. Thus, the total runtime for line 4 is $\mathcal{O}(m)$.

Line 7 can be implemented in $\mathcal{O}(nm)$ time as follows. Let $N_H(u)$ intersect the slices $S_i(s,t)$ and $S_{i+1}(s,t)$. First, update $\omega(vw')$, for all vertices $v \in S_i(s,t) \cap N_H(u)$ and all $w' \in S_{i+1}(s,t) \cap N_H(v)$. Also mark v as visited. Then, update $\omega(v'w)$ for all vertices $w \in S_{i+1}(s,t) \cap N_H(u)$ and all $v' \in S_i(s,t) \cap N_H(w)$ where v' is not marked as visited. Last, remove all marks from all vertices $v \in S_i(s,t) \cap N_H(u)$. For a given u, this runs in $\mathcal{O}(m)$ time. Thus, the total runtime for line 7 is in $\mathcal{O}(nm)$.

Finding a shortest path P such that $\omega(P)$ is maximal can be easily done in linear time. Therefore, the overall runtime of Algorithm 3 is in $\mathcal{O}(nm)$. □

Using the methods described above, we can now construct Algorithm 4 to compute a minimum eccentricity shortest path in chordal graphs.

Theorem 4. *Algorithm 4 computes a minimum eccentricity shortest path for a given chordal graph in $\mathcal{O}(n^3 m)$ time.*

Proof (Correctness). The algorithm creates a hedgehog graph $H(s,t)$ for each vertex pair s, t (line 3). Then it determines a shortest path $P(s,t)$ from s to t in $H(s,t)$ with minimal eccentricity (line 4). By Theorem 3, $P(s,t)$ is also a shortest path with minimal eccentricity from s to t in G. Therefore, for at least one pair s, t, the selected path $P(s,t)$ is a minimum eccentricity shortest path of G. Such a path is selected in line 5. □

Algorithm 4. Finds a shortest path P with minimum eccentricity for a given chordal graph G.

Input: A chordal graph $G = (V, E)$.
Output: A shortest path P with minimum eccentricity.

1 Calculate the pairwise distance for all vertices.
2 **foreach** $s, t \in V$ **do**
3   Create a hedgehog graph $H(s, t)$ of G for s and t using Algorithm 2.
4   Find a shortest path $P(s, t)$ with minimal eccentricity in $H(s, t)$ using Algorithm 3.
5 Among all shortest paths $P(s, t)$, select one for which $\text{ecc}_G(P(s, t))$ is minimal.

Proof (Complexity). Calculating the pairwise distances between vertices (line 1) can be done in $\mathcal{O}(nm)$ time. This allows to extract the distance between any two vertices in constant time. Thus, $H(s, t)$ can be created in $\mathcal{O}(n^2)$ time. By Lemma 10, finding a path with minimal eccentricity in $H(s, t)$ runs in $\mathcal{O}(nm)$ time. Therefore, the overall runtime for line 3 and line 4 is in $\mathcal{O}(n^3m)$. The total runtime for determining the eccentricities of all calculated paths to select the minimum is in $\mathcal{O}(n^2m)$. Thus, the algorithm runs in $\mathcal{O}(n^3m)$ time. □

5 Conclusion

We have investigated the Minimum Eccentricity Shortest Path problem for distance-hereditary graphs and for chordal graphs. For distance-hereditary graphs, we were able to present a linear time algorithm. For chordal graphs, we gave an $\mathcal{O}(n^3m)$ time algorithm.

The main reason for the large difference in the run-times of the two algorithms is that the second one iterates over all vertex pairs of a chordal graph. We know that, for general graphs, the problem remains NP-complete even if a start-end vertex pair is given (see the reduction in [7]). Also, we have shown that there is a shortest path with minimum eccentricity between every diametral pair of vertices of a distance-hereditary graph (Theorem 1). This leads to the following question: How hard is it to determine the start and end vertices of an optimal path? This question applies to general graphs as well as to special graph classes like chordal graphs.

Another interesting question is, for which other graph classes the problem remains NP-complete or can be solved in polynomial time. The NP-completeness proof in [7] uses a reduction from SAT. There is a planar version of 3-SAT (see [11]). Does this imply that the problem remains NP-complete for planar graphs?

Acknowledgement. This work was partially supported by the NIH grant R01 GM103309.

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