Eccentricity Approximating Trees
Extended Abstract

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Abstract. Using the characteristic property of chordal graphs that they are the intersection graphs of subtrees of a tree, Erich Prisner showed that every chordal graph admits an eccentricity 2-approximating spanning tree. That is, every chordal graph G has a spanning tree T such that $ecc_T(v) - ecc_G(v) \leq 2$ for every vertex v, where $ecc_G(v)$ ($ecc_T(v)$) is the eccentricity of a vertex v in G (in T, respectively). Using only metric properties of graphs, we extend that result to a much larger family of graphs containing among others chordal graphs and the underlying graphs of 7-systolic complexes. Furthermore, based on our approach, we propose two heuristics for constructing eccentricity k-approximating trees with small values of k for general unweighted graphs. We validate those heuristics on a set of real-world networks and demonstrate that all those networks have very good eccentricity approximating trees.

1 Introduction

All graphs $G = (V, E)$ occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. The *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest path connecting u and v in G. If no confusion arises, we will omit subindex G. The *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, it consists of all vertices (metrically) between u and v: $I(u, v) = \{x \in V : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. The *eccentricity* $ecc_G(v)$ of a vertex v in G is defined by $\max_{u \in V} d_G(u, v)$, i.e., it is the distance to a most distant vertex. The maximum value of the eccentricity represents the graph's *diameter*: $diam(G) = \max_{u \in V} ecc_G(u) = \max_{u,v \in V} d_G(u, v)$. The minimum value of the eccentricity represents the graph's *radius*: $rad(G) = \min_{u \in V} ecc_G(u)$. The set of vertices with minimum eccentricity forms the *center* $C(G)$ of a graph G, i.e., $C(G) = \{u \in V : ecc_G(u) = rad(G)\}$.

A spanning tree T of a graph G with $d_T(u, v) - d_G(u, v) \leq k$, for all $u, v \in V$, is known as an *additive tree spanner* of G [9] and, if it exists for a small integer k,

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then it gives a good approximation of all distances in G by the distances in T. Many optimization problems involving distances in graphs are known to be NP-hard in general but have efficient solutions in simpler metric spaces, with well-understood metric structures, including trees. A solution to such an optimization problem obtained for a tree spanner T of G usually serves as a good approximate solution to the problem in G.

In [13], the new notion of eccentricity approximating spanning trees was introduced by Prisner. A spanning tree T of a graph G is called an *eccentricity k-approximating spanning tree* if $ecc_T(v) - ecc_G(v) \leq k$ holds for all $v \in V$. Such a tree tries to approximately preserve only distances from each vertex v to its most distant vertices and can tolerate larger increases to nearby vertices. They are important in applications where vertices measure their degree of centrality by means of their eccentricity and would tolerate a small surplus to the actual eccentricities [13]. Note also that Nandakumar and Parthasarasthy considered in [11] eccentricity-preserving spanning trees (i.e., eccentricity 0-approximating spanning trees) and showed that a graph G has an eccentricity 0-approximating spanning tree if and only if: (a) either $diam(G) = 2rad(G)$ and $|C(G)| = 1$, or $diam(G) = 2rad(G) - 1$, $|C(G)| = 2$, and those two center vertices are adjacent; (b) every vertex $u \in V \setminus C(G)$ has a neighbor v such that $ecc_G(v) < ecc_G(u)$.

Every additive tree k-spanner is clearly eccentricity k-approximating. Therefore, eccentricity k-approximating spanning trees can be found in every interval graph for $k = 2$ [9,10,12], and in every asteroidal-triple–free graph [9], strongly chordal graph [3] and dually chordal graph [3] for $k = 3$. On the other hand, although for every k there is a chordal graph without a tree k-spanner [9,12], yet as Prisner demonstrated in [13], every chordal graph has an eccentricity 2-approximating spanning tree, i.e., with the slightly weaker concept of eccentricity-approximation, one can be successful even for chordal graphs.

Unfortunately, the method used by Prisner in [13] heavily relies on a characteristic property of chordal graphs (*chordal graphs are exactly the intersection graphs of subtrees of a tree*) and is hardly extendable to larger families of graphs.

In this paper we present a new proof of the result of [13] using only metric properties of chordal graphs (see Theorem 9 and Corollary 3). This allows us to extend the result to a much larger family of graphs which includes not only chordal graphs but also other families of graphs known from the literature.

It is known [4,15] that every chordal graph satisfies the following two metric properties:

α_1-metric: if $v \in I(u,w)$ and $w \in I(v,x)$ are adjacent, then $d_G(u,x) \geq d_G(u,v) + d_G(v,x) - 1 = d_G(u,v) + d_G(w,x)$.
triangle condition: for any three vertices u, v, w with $1 = d_G(v,w) < d_G(u,v) = d_G(u,w)$ there exists a common neighbor x of v and w such that $d_G(u,x) = d_G(u,v) - 1$.

A graph G satisfying the α_1-metric property is called an *α_1-metric graph*. If an α_1-metric graph G satisfies also the triangle condition then G is called an *(α_1, Δ)-metric graph*. We prove that every (α_1, Δ)-metric graph $G = (V, E)$ has an eccentricity 2-approximating spanning tree and that such a tree can be constructed in $\mathcal{O}(|V||E|)$ total time. As a consequence, we get that the underlying graph of every 7-systolic complex (and, hence, every chordal graph) has an eccentricity 2-approximating spanning tree.

The paper is organized as follows. In Sect. 2, we present additional notions and notations and some auxiliary results. In Sect. 3, some useful properties of the eccentricity function on (α_1, Δ)-metric graphs are described. Our eccentricity approximating spanning tree is constructed and analyzed in Sect. 4. In Sect. 5, the algorithm for the construction of an eccentricity approximating spanning tree developed in Sect. 4 for (α_1, Δ)-metric graphs is generalized and validated on some real-world networks. Our experiments show that all those real-world networks have very good eccentricity approximating trees.

Due to space limitations some proofs are omitted, they can be found in the full journal version of the paper [1].

2 Preliminaries

For a graph $G = (V, E)$, we use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and the edge set of G. We denote an *induced cycle* of length k by C_k (i.e., it has k vertices) and by W_k an *induced wheel* of size k which is a C_k with one extra vertex universal to C_k. For a vertex v of G, $N_G(v) = \{u \in V : uv \in E\}$ is called the *open neighborhood*, and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v. The distance between a vertex v and a set $S \subseteq V$ is defined as $d_G(v, S) = \min_{u \in S} d_G(u, v)$ and the set of furthest (most distant) vertices from v is denoted by $F(v) = \{u \in V : d_G(u, v) = ecc_G(v)\}$.

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if for each pair of vertices $u, v \in A$ it includes the interval $I(v, u)$ of G between u, v. An induced subgraph H of G is called *isometric* if the distance between any pair of vertices in H is the same as their distance in G. In particular, convex subgraphs are isometric. The *disk* $D(x, r)$ with center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x. In particular, the unit disk $D(x, 1) = N[x]$ comprises x and the neighborhood $N(x)$. For an edge $e = xy$ of a graph G, let $D(e, r) := D(x, r) \cup D(y, r)$.

By the definition of α_1-metric graphs clearly, such a graph cannot contain any isometric cycles of length $k > 5$ and any induced cycle of length 4. The following results characterize α_1-metric graphs and the class of chordal graphs within the class of α_1-metric graphs. Recall that a graph is *chordal* if all its induced cycles are of length 3.

Theorem 1 ([15])**.** *G is chordal if and only if it is an α_1-metric graph not containing any induced subgraphs isomorphic to cycle C_5 and wheel $W_k, k \geq 5$.*

Theorem 2 ([15])**.** *G is an α_1-metric graph if and only if all disks $D(v, k)$ ($v \in V$, $k \geq 1$) of G are convex and G does not contain the graph W_6^{++} (see Fig. 1) as an isometric subgraph.*

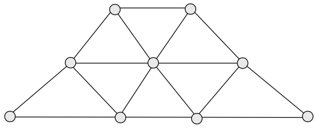

Fig. 1. Forbidden isometric subgraph W_6^{++}.

Theorem 3 ([8,14]). *All disks $D(v,k)$ ($v \in V$, $k \geq 1$) of a graph G are convex if and only if G does not contain isometric cycles of length $k > 5$, and for any two vertices x, y the neighbors of x in the interval $I(x,y)$ are pairwise adjacent.*

A graph G is called a *bridged graph* if all isometric cycles of G have length three [8]. The class of bridged graphs is a natural generalization of the class of chordal graphs. They can be characterized in the following way.

Theorem 4 ([8,14]). *$G = (V,E)$ is a bridged graph if and only if the disks $D(v,k)$ and $D(e,k)$ are convex for all $v \in V$, $e \in E$, and $k \geq 1$.*

As a consequence of Theorems 2, 3 and 4 we obtain the following equivalences.

Lemma 1. *For a graph $G = (V,E)$ the following statements are equivalent:*

(a) G is an α_1-metric graph not containing an induced C_5;
(b) G is a bridged graph not containing W_6^{++} as an isometric subgraph;
(c) The disks $D(v,k)$ and $D(e,k)$ of G are convex for all $v \in V$, $e \in E$, and $k \geq 1$, and G does not contain W_6^{++} as an isometric subgraph.

As we will show now the class of (α_1, Δ)-metric graphs contains all graphs described in Lemma 1. An induced C_5 is called *suspended* in G if there is a vertex in G which is adjacent to all vertices of the C_5.

Theorem 5. *A graph G is (α_1, Δ)-metric if and only if it is an α_1-metric graph where for each induced C_5 there is a vertex $v \in V$ such that $C_5 \subseteq N(v)$, i.e., every induced C_5 is suspended.*

We will also need the following fact.

Lemma 2. *Let $G = (V,E)$ be an (α_1, Δ)-metric graph, let K be a complete subgraph of G, and let v be a vertex of G. If for every vertex $z \in K$, $d(z,v) = k$ holds, then there is a vertex v' at distance $k-1$ from v which is adjacent to every vertex of K.*

We note here, without going into the rich theory of systolic complexes, that the underlying graph of any 7-systolic complex is nothing else than a bridged graph not containing a 6-wheel W_6 as an induced (equivalently, isometric) subgraph (see [6] for this fact and a relation of 7-systolic complexes with CAT(0) complexes). Hence, the class of (α_1, Δ)-metric graphs contains the underlying graphs of 7-systolic complexes.

3 Eccentricity Function on (α_1, Δ)-Metric Graphs

In what follows, by $C(G)$ we denote not only the set of all central vertices of G but also the subgraph of G induced by this set. We say that the eccentricity function $ecc_G(v)$ on G is *unimodal* if every vertex $u \in V \setminus C(G)$ has a neighbor v such that $ecc_G(v) < ecc_G(u)$. In other words, every local minimum of the eccentricity function $ecc_G(v)$ is a global minimum on G. It this section we will often omit subindex G since we deal only with a graph G here. A spanning tree T of G will be built only in the next section.

In this section, we will show that the eccentricity function $ecc_G(v)$ on an (α_1, Δ)-metric graph G is almost unimodal and that the radius of the center $C(G)$ of G is at most 2. Recall that for every graph G, $diam(G) \leq 2rad(G)$.

Lemma 3. *Let G be an α_1-metric graph and x be its arbitrary vertex with $ecc(x) \geq rad(G) + 1$. Then, for every vertex $z \in F(x)$ and every neighbor v of x in $I(x, z)$, $ecc(v) \leq ecc(x)$ holds.*

Proof. Assume, by way of contradiction, that $ecc(v) > ecc(x)$ and consider an arbitrary vertex $u \in F(v)$. Since x and v are adjacent, necessarily, $d(v, u) = ecc(v) = ecc(x) + 1 = d(u, x) + 1$, i.e., $x \in I(v, u)$. By the α_1-metric property, $d(u, z) \geq d(u, x) + d(v, z) = ecc(v) - 1 + ecc(x) - 1 = 2ecc(x) - 1 \geq 2rad(G) + 1$. The latter gives a contradiction to $d(u, z) \leq diam(G) \leq 2rad(G)$. □

Theorem 6. *Let G be an (α_1, Δ)-metric graph and x be an arbitrary vertex of G. If (i) $ecc(x) > rad(G) + 1$ or (ii) $ecc(x) = rad(G) + 1$ and $diam(G) < 2rad(G)$, then there must exist a neighbor v of x with $ecc(v) < ecc(x)$.*

Proof. Define for a neighbor v of x a set $S_v := \{z \in F(x) : v \in I(x, z)\}$ of vertices that are most distant from x and have v on a shortest path from x. Choose a neighbor v of x which maximizes $|S_v|$. We claim that $ecc(v) < ecc(x)$. We know, by Lemma 3, that $ecc(v) \leq ecc(x)$. Assume $ecc(v) = ecc(x)$ and consider an arbitrary vertex $u \in F(v)$.

Suppose first that $x \in I(v, u)$. Then, by the α_1-metric property, $d(u, z) \geq d(u, x) + d(v, z) = 2ecc(x) - 2$ holds for every $z \in S_v$. Hence, if $ecc(x) > rad(G) + 1$ then $d(u, z) > 2rad(G)$ and thus a contradiction to $d(u, z) \leq diam(G) \leq 2rad(G)$ arises. If, on the other hand, case (ii) applies, i.e., $ecc(x) = rad(G) + 1$ and $diam(G) < 2rad(G)$, then it follows that $d(u, z) \geq 2rad(G) > diam(G)$ and again a contradiction arises.

Now consider the case that $x \notin I(v, u)$. Then $ecc(v) = ecc(x)$ implies that $d(u, x) = d(u, v)$ and $u \in F(x)$. By the triangle condition, there must exist a common neighbor w of x and v such that $w \in I(x, u) \cap I(v, u)$. Since u belongs to S_w but not to S_v, then, by the maximality of $|S_v|$, there must exist a vertex $z \in F(x)$ which is in S_v but not in S_w. Thus, $d(w, z) > d(v, z)$ and $v \in I(w, z)$ must hold. Now, the α_1-metric property applied to $v \in I(w, z)$ and $w \in I(v, u)$ gives $d(u, z) \geq d(u, w) + d(v, z) = 2ecc(x) - 2$. As before we get $d(u, z) > 2rad(G) \geq diam(G)$, if $ecc(x) > rad(G) + 1$ (case (i)), and $d(u, z) \geq 2rad(G) > diam(G)$, if $ecc(x) = rad(G) + 1$ and $diam(G) < 2rad(G)$ (case (ii)). These contradictions complete the proof. □

For each vertex $v \in V \setminus C(G)$ of a graph G we can define a parameter $loc(v) = \min\{d(v,x) : x \in V, ecc(x) < ecc(v)\}$ and call it the *locality* of v. We define the locality of any vertex from $C(G)$ to be 1. Theorem 6 says that if a vertex v with $loc(v) > 1$ exists in an (α_1, Δ)-metric graph G then $diam(G) = 2rad(G)$ and $ecc(v) = rad(G) + 1$. That is, only in the case that $diam(G) = 2rad(G)$ the eccentricity function can be not unimodal on G.

Observe that the center $C(G)$ of a graph $G = (V, E)$ can be represented as the intersection of all the disks of G of radius $rad(G)$, i.e., $C(G) = \bigcap \{D(v, rad(G)) : v \in V\}$. Consequently, the center $C(G)$ of an α_1-metric graph G is convex (in particular, it is connected), as the intersection of convex sets is always a convex set. In general, any set $\mathcal{C}_{\leq i}(G) := \{z \in V : ecc(z) \leq rad(G) + i\}$ is a convex set of G as $\mathcal{C}_{\leq i}(G) = \bigcap \{D(v, rad(G) + i) : v \in V\}$.

Corollary 1. *In an α_1-metric graph G, all sets $\mathcal{C}_{\leq i}(G)$, $i \in \{0, \ldots, diam(G) - rad(G)\}$, are convex. In particular, $C(G)$ of an α_1-metric graph G is convex.*

The following result gives bounds on the diameter and the radius of the center of an (α_1, Δ)-metric graph. Previously it was known that the diameter (the radius) of the center of a chordal graph is at most 3 (at most 2, respectively) [5].

Theorem 7. *For an (α_1, Δ)-metric graph G, $rad(C(G)) \leq 2$.*

Proof. Assume, by way of contradiction, that there are vertices $s, t \in C(G)$ such that $d(s,t) = 4$. Consider an arbitrary shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$. Since $C(G)$ is convex any shortest path connecting s and t is in $C(G)$.

First we claim that for any vertex $u \in F(x_3)$ all vertices of P are at distance $r := d(u, x_3) = rad(G)$ from u. As $x_i \in C(G)$, we know that $d(u, x_i) \leq r$ $(1 \leq i \leq 5)$. Assume $d(u, x_i) = r - 1$, $d(u, x_{i+1}) = r$, and $i \leq 2$. Then, the α_1-metric property applied to $x_i \in I(u, x_{i+1})$ and $x_{i+1} \in I(x_i, x_{i+3})$ gives $d(x_{i+3}, u) \geq r - 1 + 2 = r + 1$ which is a contradiction to $d(u, x_{i+3}) \leq r$. So, $d(u, x_1) = d(u, x_2) = r$. By symmetry, also $d(u, x_4) = d(u, x_5) = r$.

By the triangle condition, there must exist vertices v and w at distance $r - 1$ from u such that $vx_1, vx_2, wx_4, wx_5 \in E$. We claim that x_3 is adjacent to neither v nor w. Assume, without loss of generality, that $vx_3 \in E$. Then, $d(x_5, x_1) = 4$ implies $d(x_5, v) = 3$ and therefore $x_3 \in I(x_5, v)$. Now, the α_1-metric property applied to $x_3 \in I(x_5, v)$ and $v \in I(u, x_3)$ gives $d(x_5, u) \geq r - 1 + 2 = r + 1$ which is impossible. So, $vx_3, wx_3 \notin E$.

Obviously, $vx_4, wx_2 \notin E$. If $d(x_4, v) = 3$ then $x_2 \in I(x_4, v)$. Thus, by $v \in I(x_2, u)$ and the α_1-metric property, we would get $d(x_4, u) \geq r - 1 + 2 = r + 1$ which, again, is impossible. Thus, $d(x_4, v) = 2$ must hold. Since, by Theorem 5, every induced C_5 is suspended in G and, further, G cannot contain an induced C_4, we can choose a vertex $y \in N(v) \cap N(x_4)$ which is adjacent both to x_2 and x_3 as well. If $d(y, u) = r$ then again $y \in I(v, x_5)$ and $v \in I(u, y)$ will imply $d(x_5, u) \geq r - 1 + 2 = r + 1$, which is impossible. So, $d(y, u) = r - 1$ must hold and, by the convexity of disks, y must be adjacent to w.

All the above holds for every shortest path $P = (s = x_1, x_2, x_3, x_4, x_5 = t)$ connecting vertices s and t. Now, assume that P is chosen in such a way that among all vertices in $I(s,t)$ that are at distance 2 from s (we will call this set of vertices $S_2(s,t)$) the vertex x_3 has the minimum number of furthest vertices, i.e., $|F(x_3)|$ is as small as possible. Observe that, by convexity of the center, $S_2(s,t) \subseteq C(G)$. As y also belongs to $S_2(s,t)$ and has u at distance $r-1$, by the choice of x_3, there must exist a vertex $u' \in F(y)$ which is at distance $r-1$ from x_3. Applying the previous arguments to the path $P' := (s = x_1, x_2, y, x_4, x_5 = t)$, we will have $d(x_i, u') = d(y, u') = r$ for $i = 1, 2, 4, 5$, and get two more vertices v' and w' at distance $r-1$ from u' such that $v'x_1, v'x_2, w'x_4, w'x_5 \in E$ and $v'y, w'y \notin E$. By the convexity of disk $D(u', r-1)$, also $v'x_3, w'x_3 \in E$. Now consider the disk $D(x_2, 2)$. Since w, w' are in the disk and x_5 is not, vertices w and w' must be adjacent. But then vertices y, x_3, w', w form a forbidden induced cycle C_4.

The obtained contradictions show that a shortest path P of length 4 cannot exist in $C(G)$, i.e., $diam(C(G)) \leq 3$. As $C(G)$ is a convex set of G, the subgraph of G induced by $C(G)$ is also an α_1-metric graph. According to [15], $diam(G) \geq 2rad(G) - 2$ holds for every α_1-metric graph G. Hence, for a graph induced by $C(G)$ we have $3 \geq diam(C(G)) \geq 2rad(C(G)) - 2$, i.e., $rad(C(G)) \leq 2$. □

Corollary 2 ([5]). *For a chordal graph G, $rad(C(G)) \leq 2$.*

For our next arguments we need a generalization of the set $S_2(s,t)$, as used in the proof of Theorem 7. We define a *slice* of the interval $I(u,v)$ from u to v for $0 \leq k \leq d(u,v)$ to be the set $S_k(u,v) = \{w \in I(u,v) : d(w,u) = k\}$.

Theorem 8. *Let G be an (α_1, Δ)-metric graph. Then, in every slice $S_k(u,v)$ there is a vertex x that is universal to that slice, i.e., $S_k(u,v) \subseteq N[x]$. In particular, if $diam(G) = 2rad(G)$, then $diam(C(G)) \leq 2$ and $rad(C(G)) \leq 1$.*

4 Eccentricity Approximating Spanning Tree Construction

It this section, we construct an eccentricity approximating spanning tree and analyze its quality for (α_1, Δ)-metric graphs. Here, we will use sub-indices G and T to indicate whether the distances or the eccentricities are considered in G or in T. $I(u,v)$ will always mean the interval between vertices u and v in G.

4.1 Tree Construction for Unimodal Eccentricity Functions

First consider the case when the eccentricity function on G is unimodal, that is, every non-central vertex of G has a neighbor with smaller eccentricity. We will need the following lemmas.

Lemma 4 ([7]). *Let G be an arbitrary graph. The eccentricity function on G is unimodal if and only if, for every vertex v of G, $ecc_G(v) = d_G(v, C(G)) + rad(G)$.*

Lemma 5 ([2]). *Let G be an arbitrary α_1-metric graph. Let x, y, v, u be vertices of G such that $v \in I(x, y)$, $x \in I(v, u)$, and x and v are adjacent. Then $d(u, y) = d(u, x) + d(v, y)$ holds if and only if there exist a neighbor x' of x in $I(x, u)$ and a neighbor v' of v in $I(v, y)$ with $d_G(x', v') = 2$; in particular, x' and v' lie on a common shortest path of G between u and y.*

We construct a spanning tree T of G as follows. First find the center $C(G)$ of G and pick an arbitrary central vertex c of the graph $C(G)$, i.e., $c \in C(C(G))$. Compute a *breadth-first-search tree* T' of $C(G)$ started at c. Expand this tree T' to a spanning tree T of G by identifying for every vertex $v \in V \setminus C(G)$ its parent vertex in the following way: among all neighbors x of v with $ecc_G(x) = ecc_G(v) - 1$ pick that vertex which is closest to c in G (break ties arbitrarily).

Lemma 6. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is unimodal. Then, for a tree T constructed as described above and every vertex v of G, $d_G(v, c) = d_T(v, c)$ holds, i.e., T is a shortest-path-tree of G started at c.*

Proof. Let v be an arbitrary vertex of G and let v' be a vertex of $C(G)$ closest to v in T. By Lemma 4 and by the construction of T, $d_G(v, v') = d_T(v, v')$ and v' is a vertex of $C(G)$ closest to v in G. By the construction of T', also $d_G(c, v') = d_T(c, v')$ (note that, as $C(G)$ is a convex subgraph of G, clearly, $d_{C(G)}(x, y) = d_G(x, y)$ for every pair x, y of $C(G)$). So, in the tree T, we have $d_T(c, v') + d_T(v', v) = d_T(v, c)$. If $d_G(c, v') + d_G(v', v) = d_G(v, c)$, then $d_G(v, c) = d_T(v, c)$, and we are done. Assume, therefore, that $d_G(c, v') + d_G(v', v) > d_G(v, c)$ and among all vertices that fulfill this inequality, let v be the one that is closest to $C(G)$. Consider the neighbor x of v' on the path in T from v' to v. We have $x \in I(v', v)$ and, by Lemma 4, $ecc_G(x) = rad(G) + 1$. Note that $x = v$ is possible.

If $v' \notin I(x, c)$ then $d_G(x, c) \leq d_G(v', c)$. By the convexity of $C(G)$, x with $ecc_G(x) = rad(G) + 1$ cannot be on a shortest path between two central vertices c and v'. Hence, $d_G(x, c) = d_G(v', c)$ holds. By the triangle condition, there must exist a common neighbor y of v' and x which is at distance $d_G(v', c) - 1$ from c. Since $y \in I(v', c)$, by the convexity of $C(G)$, $ecc_G(y) = rad(G)$. But then, as y is closer to c than v' is, vertex x cannot choose v' as its parent in T, since y is a better choice.

If $v' \in I(x, c)$ then, by the α_1-metric property, $d_G(c, v') + d_G(x, v) \leq d_G(v, c)$. As $d_G(c, v') + d_G(v', v) > d_G(v, c)$, we have $d_G(c, v') + d_G(x, v) = d_G(v, c)$. By Lemma 5, there must exist a neighbor x' of x in $I(x, v)$ and a neighbor v'' of v' in $I(v', c)$ with $d_G(x', v'') = 2$. Denote by w a common neighbor of x' and v''. We have $d_G(x, c) > d_G(w, c)$. Set $k := d_G(v, v') = d_G(v, C(G)) = ecc_G(v) - rad(G)$. Let $P_T := (x = a_1, \ldots, a_k = v)$ be the path in T between x and v. Let $P_G := (w = b_1, x' = b_2, \ldots, b_k = v)$ be a shortest path of G between w and v which shares a longest suffix with P_T, that is, $a_j = b_j$ for all $j > i$, $a_i \neq b_i$, and i is minimal under these conditions. Note that $i = 1$ and $a_2 = b_2 = v$ is possible. By Lemma 4, $ecc_G(a_i) = ecc_G(b_i) = rad(G) + i = ecc_G(a_{i+1}) - 1$.

Since v is a vertex closest to $C(G)$ fulfilling inequality $d_G(c, v') + d_G(v', v) > d_G(v, c)$, for vertex a_i ($i < k$), the equation $d_G(c, v') + d_G(v', a_i) = d_G(a_i, c)$ holds. Hence, $d_G(c, x) + d_G(x, a_i) = d_G(a_i, c)$. Also, by Lemma 5,

$d_G(c, w) + d_G(w, b_i) = d_G(b_i, c)$. Consequently, $d_G(x, c) > d_G(w, c)$ and $d_G(x, a_i) = d_G(w, b_i)$ imply $d_G(a_i, c) > d_G(b_i, c)$. Therefore, vertex a_{i+1} cannot choose a_i as its parent in T, since b_i is a better choice.

The obtained contradictions prove that $d_G(c, v') + d_G(v', v) = d_G(v, c)$ and hence $d_G(v, c) = d_T(v, c)$. □

Lemma 7. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is unimodal. Then, for a tree T constructed as described above and for every vertex v of G, $ecc_T(v) \leq ecc_G(v) + rad(C(G))$ holds.*

Proof. Let v be an arbitrary vertex of G, v' be a vertex of $C(G)$ closest to v in T, and u be a vertex most distant from v in T, i.e., $ecc_T(v) = d_T(v, u)$. By Lemma 4 and by the construction of T, $d_G(v, v') = d_T(v, v')$ and v' is a vertex of $C(G)$ closest to v in G. We have $ecc_T(v) = d_T(v, u) \leq d_T(v, v') + d_T(v', c) + d_T(c, u)$, where $c \in C(C(G))$ is the root of the tree T (see the construction of T). Since $d_G(v, v') = d_T(v, v')$, $d_T(v', c) = d_G(v', c) \leq rad(C(G))$, and $d_T(c, u) = d_G(c, u) \leq rad(G)$ (by Lemma 6 and the fact that $c \in C(C(G))$), we obtain $ecc_T(v) \leq d_G(v, v') + rad(C(G)) + rad(G) = ecc_G(v) + rad(C(G))$, as $d_G(v, v') + rad(G) = d_G(v, C(G)) + rad(G) = ecc_G(v)$ by Lemma 4. □

4.2 Construction for Eccentricity Functions that Are Not Unimodal

Consider now the case when the eccentricity function on G is not unimodal, that is, there is at least one vertex $v \notin C(G)$ in G which has no neighbor with smaller eccentricity. By Theorem 6, $ecc_G(v) = rad(G) + 1$, $diam(G) = 2rad(G)$ and v has a neighbor with the eccentricity equal to $ecc_G(v)$. We will need the following weaker version of Lemma 4.

Lemma 8. *Let $G = (V, E)$ be an (α_1, Δ)-metric graph. Let v be an arbitrary vertex of G and v' be an arbitrary vertex of $C(G)$ closest to v. Then,*

$$d_G(v, C(G)) + rad(G) - 1 \leq ecc_G(v) \leq d_G(v, C(G)) + rad(G).$$

Furthermore, there is a shortest path $P := (v' = x_0, x_1, \ldots, x_\ell = v)$, connecting v with v', for which the following holds:

(a) if $ecc_G(v) = d_G(v, C(G)) + rad(G)$ then $ecc_G(x_i) = d_G(x_i, C(G)) + rad(G) = i + rad(G)$ for each $i \in \{0, \ldots, \ell\}$;
(b) if $ecc_G(v) = d_G(v, C(G)) + rad(G) - 1$ then $ecc_G(x_i) = d_G(x_i, C(G)) - 1 + rad(G) = i - 1 + rad(G)$ for each $i \in \{3, \ldots, \ell\}$ and $ecc_G(x_1) = ecc_G(x_2) = rad(G) + 1$.

In particular, if $ecc_G(v) = rad(G) + 1$ then $d_G(v, C(G)) \leq 2$.

Now we are ready to construct an eccentricity approximating spanning tree T of G for the case when the eccentricity function is not unimodal. We know that $diam(G) = 2rad(G)$ in this case and, therefore, $C(G) \subseteq S_{rad(G)}(x, y)$ for any diametral pair of vertices x and y, i.e., for x, y with $d_G(x, y) = diam(G)$.

By Theorem 8 and since $C(G)$ is convex, there is a vertex $c \in C(G)$ such that $C(G) \subseteq N[c]$. First we find such a vertex c in $C(G)$ and build a tree T' by making c adjacent with every other vertex of $C(G)$. Then, we expand this tree T' to a spanning tree T of G by identifying for every vertex $v \in V \setminus C(G)$ its parent vertex in the following way: if v has a neighbor with eccentricity less than $ecc_G(v)$, then among all such neighbors pick that vertex which is closest to c in G (break ties arbitrarily); if v has no neighbors with eccentricity less than $ecc_G(v)$ (i.e., $ecc_G(v) = rad(G) + 1$ by Theorem 6), then among all neighbors x of v with $ecc_G(x) = ecc_G(v) = rad(G)+1$ pick again that vertex which is closest to c in G (break ties arbitrarily).

Lemma 9. *Let G be an (α_1, Δ)-metric graph whose eccentricity function is not unimodal. Then, for a tree T constructed as described above and every vertex v of G, $d_T(v, c) = d_G(v, c)$ holds.*

Proof. Assume, by way of contradiction, that $d_G(v, c) < k := d_T(v, c)$ and let v be a vertex with such a condition that has smallest eccentricity $ecc_G(v)$. We may assume that $ecc_G(v) > rad(G) + 1$. Indeed, every v with $ecc_G(v) = rad(G) + 1$ either has a neighbor in $C(G)$ or has a neighbor with a neighbor in $C(G)$ (see Lemma 8). Therefore, if $d_G(v, c) < d_T(v, c)$ then, by the construction of T, necessarily $d_G(v, c) = 2$, $d_T(v, c) = 3$ and the neighbor x of v on the path of T between v and c must have the eccentricity equal to $rad(G) + 1 = ecc_G(v)$. But then, for a common neighbor w of v and c in G, $ecc_G(w) \leq rad(G) + 1$ must hold and hence vertex v cannot choose x as its parent in T, since w is a better choice.

So, let $ecc_G(v) > rad(G) + 1$. By Lemma 8, there must exist a shortest path in G between v and c such that the neighbor w of v on this path has eccentricity $ecc_G(w) = ecc_G(v) - 1$. Hence, by the construction of T, $ecc_G(x) = ecc_G(v) - 1$ must hold for the neighbor x of v on the path of T between v and c. By the minimality of $ecc_G(v)$, we have $d_G(x, c) = d_T(x, c) = k - 1$. Since $d_G(w, c) = d_G(v, c) - 1 < k - 1$, a contradiction arises; again v cannot choose x as its parent in T, since w is a better choice. □

Lemma 10. *Let G be an (α_1, Δ)-metric graph with $diam(G) = 2rad(G)$. Then, for a tree T constructed as described above and every vertex v of G, $ecc_T(v) \leq ecc_G(v) + 2$ holds.*

Proof. Let v be an arbitrary vertex of G and u be a vertex most distant from v in T, i.e., $ecc_T(v) = d_T(v, u)$. We have $ecc_T(v) = d_T(v, u) \leq d_T(v, c) + d_T(c, u) = d_G(v, c) + d_G(c, u) \leq d_G(v, c) + rad(G) \leq d_G(v, C(G)) + 1 + rad(G) \leq ecc_G(v) + 2$ since $d_G(c, u) \leq ecc_G(c) = rad(G)$, $d_G(v, c) \leq d_G(v, C(G)) + 1$ (recall that $C(G) \subseteq N[c]$), and $d_G(v, C(G)) - 1 + rad(G) \leq ecc_G(v)$ (by Lemma 8). □

Our main result is the following theorem. It combines Theorem 7, Lemmas 7 and 10; the complexity follows straightforward.

Theorem 9. *Every (α_1, Δ)-metric graph $G = (V, E)$ has an eccentricity 2-approximating spanning tree. Furthermore, such a tree can be constructed in $\mathcal{O}(|V||E|)$ total time.*

As a consequence we have the following corollary. Note that the result of Corollary 3 (and hence of Theorem 9) is sharp as there are chordal graphs that do not have any eccentricity 1-approximating spanning tree [13].

Corollary 3. *The underlying graph of every 7-systolic complex has an eccentricity 2-approximating spanning tree. In particular, every chordal graph has an eccentricity 2-approximating spanning tree.*

5 Experimental Results for Some Real-World Networks

We say that a tree T is an *eccentricity k-approximating tree* for a graph G if for every vertex v of G, $|ecc_T(v) - ecc_G(v)| \leq k$ holds. If T is a spanning tree, then $ecc_T(v) \geq ecc_G(v)$, for all $v \in V$, and this new definition agrees with the definition of an eccentricity k-approximating *spanning* tree.

Table 1. A spanning tree T constructed by heuristic EAST: for each vertex $u \in V$, $k(u) = ecc_T(u) - ecc_G(u)$; $k_{max} = \max_{u \in V} k(u)$; $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$. A tree T' constructed by heuristic EAT: for each vertex $u \in V$, $k(u) = ecc_{T'}(u) - ecc_G(u)$; $k_{max} = \max_{u \in V} k(u)$; $k_{min} = \min_{u \in V} k(u)$; $k_{avg} = \frac{1}{n} \sum_{u \in V} k(u)$

Network	$diam(G)$	k_{max} of T	k_{avg} of T	$[k_{min}, k_{max}]$ of T'	k_{avg} of T'
EMAIL	8	3	1.774	$[-1, 0]$	-0.0009
FACEBOOK	8	2	0.69	$[0, 0]$	0
DUTCH-ELITE	22	6	2.083	$[-1, 0]$	-0.771
JAZZ	6	2	1.742	$[-1, 0]$	-0.015
EVA	18	2	0.575	$[-1, 0]$	-0.36
AS-GRAPH-1	9	2	0.64	$[0, 1]$	0.62
AS-GRAPH-2	11	3	1.272	$[0, 1]$	0.949
AS-GRAPH-3	9	2	0.312	$[0, 1]$	0.248
E-COLI-PI	5	2	0.769	$[0, 1]$	0.595
YEAST-PI	12	4	0.972	$[-1, 0]$	-0.168
MACAQUE-BRAIN-1	4	1	0.222	$[0, 0]$	0
MACAQUE-BRAIN-2	4	2	1.489	$[-1, 0]$	-0.003
E-COLI-METABOLIC	16	4	1.132	$[-1, 0]$	-0.624
C-ELEGANS-METABOLIC	7	1	0.349	$[0, 1]$	0.342
YEAST-TRANSCRIPTION	9	3	1.121	$[0, 1]$	0.019
US-AIRLINES	6	0	0	$[0, 0]$	0
POWER-GRID	46	4	1.409	$[-3, 0]$	-1.309
WORD-ADJACENCY	5	1	0.411	$[0, 1]$	0.152
FOOD	4	2	1.629	$[-1, 0]$	-0.015

Based on what we learned from (α_1, Δ)-metric graphs in Sect. 4, we propose two heuristics for constructing eccentricity approximating trees in general graphs and analyze their performance on a set of real-world networks. Both heuristics try to mimic the construction for (α_1, Δ)-metric graphs that we used in Sect. 4. For more details on the data-set and the experiments see the full journal version of the paper [1].

Our first heuristic, named *EAST*, constructs an *E*ccetricity *A*pproximating *S*panning *T*ree T_{EAST} as a shortest-path-tree starting at a vertex $c \in C(C(G))$. We identify an arbitrary vertex $c \in C(C(G))$ as the root of T_{EAST}, and for each other vertex v of G define its parent in T_{EAST} as follows: among all neighbors of v in $I(v, c)$ choose a vertex with minimum eccentricity (break ties arbitrarily).

Our second heuristic, named *EAT*, constructs for a graph G an *E*ccetricity *A*pproximating *T*ree T_{EAT} (not necessarily a spanning tree; it may have a few edges not present in graph G) as follows. We again identify an arbitrary vertex $c \in C(C(G))$ as the root of T_{EAT} and make it adjacent in T_{EAT} to all other vertices of $C(G)$ (clearly, some of these edges might not be contained in G). Then, for each vertex $v \in V \setminus C(G)$, we find a vertex u with $ecc_G(u) < ecc_G(v)$ which is closest to v, and if there is more than one such vertex, we pick the one which is closest to c. In other words, among all vertices $\{u \in V : d_G(u, v) = loc(v)$ and $ecc_G(u) < ecc_G(v)\}$, we choose a vertex u which is closest to c (break ties arbitrarily). Such a vertex u becomes the parent of v in T_{EAT}. Clearly, if $loc(v) > 1$ then edge uv of T_{EAT} is not present in G.

We tested both heuristics on a set of real-world networks. Experimental results obtained are presented in Table 1. See the full journal version of the paper [1] for more details. It turns out that the eccentricity terrain of each of those networks resembles the eccentricity terrain of a tree.

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