On Greedy Matching Ordering and Greedy Matchable Graphs *

(Extended Abstract)

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Abstract. In this note a greedy algorithm is considered that computes a matching for a graph with a given ordering of its vertices, and those graphs are studied for which a vertex ordering exists such that the greedy algorithm always yields maximum cardinality matchings for each induced subgraph. We show that these graphs, called greedy matchable graphs, are a subclass of weakly triangulated graphs and contain strongly chordal graphs and chordal bipartite graphs as proper subclasses. The question when can this ordering be produced efficiently is discussed too.

1 Introduction and The Greedy Algorithm

Throughout this note all graphs $G = (V, E)$ are finite, undirected and simple (i.e. without loops and multiple edges). The *(open) neighborhood* of a vertex v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$.

A *matching* of a graph G is a subset M of E such that no two edges share a vertex. A matching of maximal size is called a *maximum matching*. The problem of constructing maximum matchings is one of the most extensively studied problems of graph theory and this is mostly due to the wide variety of applications (see [17, 20]).

Assume that we are given a graph with an ordering on vertices $\sigma = (v_1, v_2, \ldots, v_n)$ (in practice, when we are given a graph, usually it is given with ordered vertex set) and we want to solve the maximum matching problem. One of the simplest way to construct a matching is to choose the smallest (with respect to σ) neighbor of the first vertex to match this vertex, and continue in a similar fashion stepping through the ordering: for each unmatched vertex, we encounter, choose its smallest unmatched neighbor if there is one. A formal description of this method is presented below.

Algorithm (greedy matching)

Input: A graph G with an ordering of vertices $\sigma = (v_1, v_2, \ldots, v_n)$.
Output: A matching M of G.

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Complexity: $O(|V| + |E|)$.
Method:

$V' := V$; $M := \emptyset$;
for $i := 1$ **to** n **do**
  if $v_i \in V'$ and $N(v_i) \cap V' \neq \emptyset$ **then**
    choose a neighbor $u \in V'$ of v_i which is smallest with respect to σ;
    add the edge $v_i u$ to M;
    delete the vertices v_i and u from V'

Such a strategy is used by QUEYRANNE ET AL. [21] to solve special transport problems and by DAHLHAUS and KARPINSKI [6] to compute a maximum matching for strongly chordal graphs. A graph G is called *strongly chordal* if it admits a *strong simplicial ordering* [8], i.e. an ordering σ of the vertices of G such that

1. if $a < \{b, c\}$ and $ab, ac \in E$ then $bc \in E$,
2. if $ab, ac, bd \in E$, $a < d$ and $b < c$ then $cd \in E$.

(We write $a < b$ whenever in a given ordering σ vertex a has a smaller number than vertex b.) DAHLHAUS and KARPINSKI have shown in [6] that along a strong simplicial ordering the greedy matching algorithm always yields a maximum matching. Moreover, this ordering allows to compute a maximum matching for every induced subgraph of a strongly chordal graph, i.e. to solve a kind of constrained matching problem, when we want to find a maximum matching M of a graph G such that both end vertices of each edge in M are from a given set $S \subseteq V$. Unfortunately, to date, there are no linear algorithms to give a strong simplicial ordering of a strongly chordal graph $G = (V, E)$: the fastest such algorithm takes $O(|E| \log |V|)$ [19] or $O(|V|^2)$ [23] time. When a strong simplicial ordering is given then the maximum matching problem is solved in linear time.

In this note we introduce and study the graphs for which a vertex ordering exists such that the greedy matching algorithm always gives maximum matchings for each induced subgraph.

Definition 1. An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the vertex set of a graph $G = (V, E)$ is a *greedy matching ordering* if, for each induced subgraph F of G with ordering σ_F induced by σ, the matching computed by the above algorithm for F is a maximum matching.

Definition 2. A graph G is *greedy matchable* if it admits a greedy matching ordering.

We give a characterization of greedy matching orderings in terms of forbidden induced suborderings and present some properties of greedy matchable graphs. We show that these graphs are a subclass of weakly triangulated graphs and contain both strongly chordal graphs and chordal bipartite graphs as proper subclasses. Moreover, not only strongly chordal graphs and chordal bipartite graphs are greedy matchable but all graphs which admit an ordering satisfying the second condition of strong simplicial orderings. In the last section we characterize those greedy matchable graphs for which a greedy matching ordering can be produced efficiently by Lexicographic Breadth–First–Search or lexical ordering.

2 Greedy Matching Ordering

Here we characterize greedy matching orderings by forbidden induced suborders and show that greedy matching orderings can be recognized in $O(|V||E|)$ time.

In what follows the expression $\{a, b\} < \{c, d\}$ will mean that in a given ordering σ both vertices a and b have smaller numbers than vertices c and d.

Definition 3. *An ordering σ of the vertex set of a graph G is admissible if for every four vertices a, b, c, d of G such that $ab, ac, bd \in E$ the following holds.*

(i) If $\{a, b\} < \{c, d\}$ then $cd \in E$.
(ii) If $a < d < b < c$ and $ad \notin E$ then $cd \in E$.

In other words, σ is admissible for G if G ordered with respect to σ does not contain induced ordered subgraphs listed below.

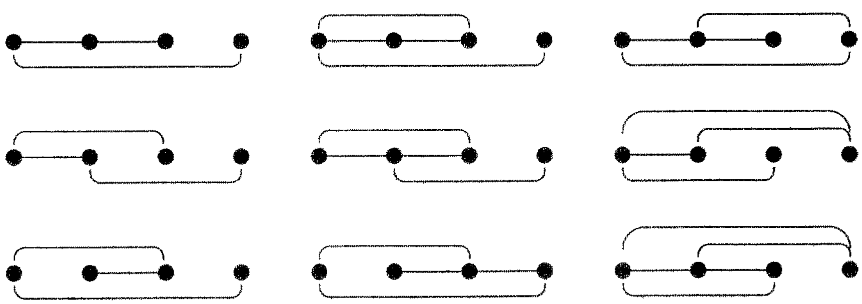

FIGURE 1. Forbidden ordered subgraphs.

Theorem 1. *An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the vertex set of a graph G is a greedy matching ordering if and only if it is admissible.*

Proof. It is easy to see that, for ordered graphs presented in Figure 1, the greedy matching algorithm will give a matching of cardinality one while all these graphs have matching consisting of two edges. Hence, any greedy matching ordering of G must be admissible.

The converse it is sufficient to prove only for graph G itself; note that an ordering σ, admissible for G, is admissible for every induced subgraph of G as well.

For a given matching M of G, we denote by $M_{\leq x}$ the matching restricted to the set $\{v \in V : v \leq x\}$. Let M^* be an arbitrary maximum matching of G and

M' be the matching of G computed by the above algorithm. We will show that it is possible to transform M^* to another maximum matching that coincides with M'. Consider the smallest in σ vertex x such that $M^*_{\leq x}$ and $M'_{\leq x}$ are different. Then $M^*_{<x}$ and $M'_{<x}$ coincide. We distinguish between three cases.

Case 1. There is an edge xt' in $M'_{\leq x}$ but no edge incident to x in $M^*_{\leq x}$.
If vertex t' is not incident to any edge from M^* then in M^* we can replace the edge incident to x by xt' (since M^* is a maximum matching such an edge xt with $t > x$ exists). Analogously, if vertex x is not incident to any edge from M^* then in M^* we can replace the edge incident to t' by xt'. Hence, we may assume that there are two edges in M^* of the form $t'y$ and xt. Moreover, since $M^*_{<x}$ and $M'_{<x}$ coincide and x has no incident edges in $M^*_{\leq x}$, we have $x < y$ and $x < t$.
Thus, the vertices t', x, y, t with $t'x$,$t'y$,$xt \in E$ fulfill the condition $t' < x < \{y, t\}$. So far as σ is an admissible ordering of G, the vertices y and t must be adjacent. But now we can replace in M^* xt and $t'y$ by $t'x$ and ty.

Case 2. There is an edge xt in $M^*_{\leq x}$ but no edge incident to x in $M'_{\leq x}$.
Since there is no edge in $M'_{\leq x}$ incident to x on step when the vertex t is considered by algorithm either t is not in V' or there is in V' a neighbor of t smaller than x. In both cases vertex t is incident to an edge from $M'_{<x}$. This is a contradiction to $M^*_{<x} = M'_{<x}$.

Case 3. There exists an edge xt in $M^*_{\leq x}$ and an edge xt' in $M'_{\leq x}$.
Assume that $t < t'$. Since $xt \notin M^T$ and x belongs to V' on step when the vertex t is considered by algorithm there exists a vertex $y < x$ such that $ty \in M'$. But this contradicts $M^*_{<x} = M'_{<x}$. Therefore, $t' < t < x$.
If vertex t' is not incident to any edge from M^* then in M^* we can replace the edge xt by xt'. So, assume that there is an edge $t'y \in M^*$. From $M^*_{<x} = M'_{<x}$ we conclude that $y > x$.
Suppose now that the vertices t' and t are adjacent. Since $t < x$ and $t'x$ belongs to M', by algorithm the vertex t is not in V' on the step when the vertex t' is considered. This means that there exists a vertex $a < t'$ such that $at \in M'$. Again a contradiction to $M^*_{<x} = M'_{<x}$ arises (recall that $tx \in M^*$). Hence, the vertices t' and t are not adjacent. Additionally we had $t' < t < x < y$ and $t'x$,$t'y$,$xt \in E$. Since σ is an admissible ordering, the vertices y and t must be adjacent in G. As before we can replace in M^* xt and $t'y$ by $t'x$ and ty.

Thus, we have shown how to transform M^* to a new maximum matching that coincides with M' in $\{v \in V : v \leq x\}$. By induction we get a maximum matching that coincides with M' in V. □

From Theorem 1 we can derive that a graph G is greedy matchable if and only if there is an ordering of V which contains no suborderings listed in Figure 1.

Recall that an ordering σ is a strong simplicial ordering if the following two conditions hold.

1. If $a < \{b, c\}$ and $ab, ac \in E$ then $bc \in E$.
2. If $ab, ac, bd \in E$, $a < d$ and $b < c$ then $cd \in E$.

If an ordering σ satisfies only the first condition then σ is called *simplicial ordering*. It is well-known that a graph G has a simplicial ordering if and only if G is *chordal*, i.e. it has no induced cycles of length greater than three (cf. [9]). An ordering satisfying only the second condition we will call *strong ordering*.

Definition 4. *An ordering σ of the vertex set of a graph G is a strong ordering if for every four vertices a, b, c, d of G such that $ab, ac, bd \in E$, $a < d$ and $b < c$ we have $cd \in E$.*

An immediate consequence of the above theorem is the following.

Corollary 1. *Every strong ordering is a greedy matching ordering.*

Hence, all graphs which admit a strong ordering are greedy matchable graphs. Among these graphs are also the well-known class of *chordal bipartite graphs*, i.e. bipartite graphs having no induced cycles of length greater than four (cf. [9]). It follows from the results in [8, 1, 18] that chordal bipartite graphs are exactly those bipartite graphs that have a strong ordering.

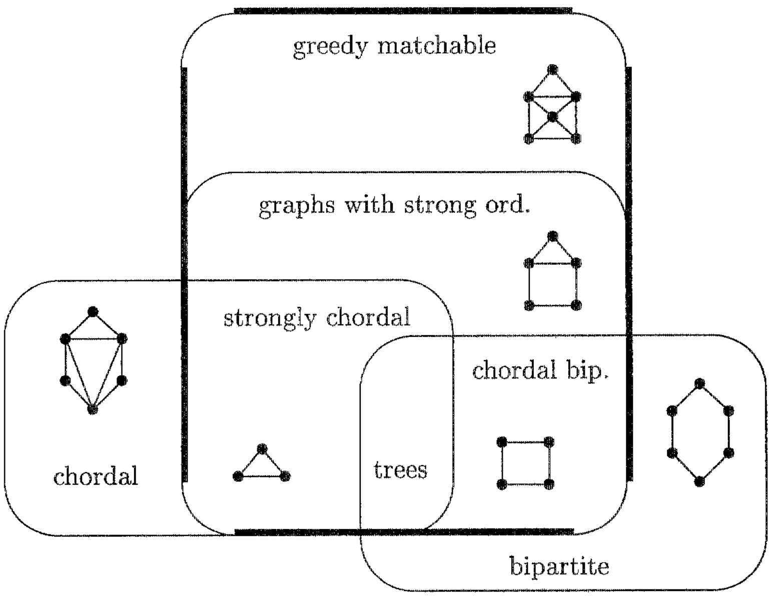

FIGURE 2. Some subclasses of greedy matchable graphs.

The next result indicates how we can check a given ordering for admissibility.

Theorem 2. *An ordering σ of the vertex set of a graph G is admissible if and only if the following holds.*

(*i*) *If* $ad \in E$, $a < d$ *and there exists a vertex* $b \in N(a) \cap N(d)$ *such that* $a < b < d$ *then* $cd \in E$ *for every* $c \in N(a)$, $c > d$.
(*ii*) *If* $ab \in E$, $a < b$ *and* c *is the smallest neighbor of* a *with respect to* σ *satisfying* $c > b$ *then* $cd \in E$ *for every vertex* $d \in N(b)$ *such that* $d < c$ *and* $ad \notin E$ *or* $d > c$.

Proof is omitted.

Since both these conditions can be verified in $O(|V||E|)$ time, it can be decided in $O(|V||E|)$ time whether a given ordering σ is a greedy matching ordering.

3 Greedy Matchable Graphs

By P_k and C_k we denote a path and a cycle on k vertices. An induced cycle C_k with $k \geq 5$ is called a *hole*. An *antihole* is the complement of a hole. We call a graph *nontrivial* if it has more than one vertex.

For a given ordering σ of the vertex set of a graph G, by $G_{\geq v}$ we denote a subgraph of G induced by the set $\{u \in V : u \geq v\}$. A vertex v of a graph G is called *simplicial* if every two neighbors of v are adjacent, i.e. the neighborhood $N(v)$ of v induces a complete subgraph of G. Using this notions one can give an alternative definition of simplicial orderings. An ordering σ is a *simplicial ordering* of G if vertex v is simplicial in $G_{\geq v}$ for all $v \in V$. In [10] the notion of simpliciality was adapted for bipartite graphs. An edge ab of a bipartite graph G is called *bisimplicial* if $N(a) \cup N(b)$ induces a complete bipartite subgraph of G. Analogously to this we define a simplicial edge of an arbitrary graph.

Definition 5. *An edge* ab *of a graph* G *is simplicial if every two vertices* $v \in N(a)$ *and* $u \in N(b)$ $(u \neq v)$ *are adjacent in* G.

Lemma 1. *Every connected nontrivial induced subgraph of a greedy matchable graph has a simplicial edge.*

Proof. Let σ be an admissible ordering of a graph G and $F = (V', E')$ be an arbitrary connected nontrivial induced subgraph of G. Consider leftmost vertices $a \in V'$ and $b \in N(a) \cap V'$ with respect to σ. From the choice of a and b we have $a < d$ and $a < b < c$ for all vertices $c \in N(a) \cap V'$ and $d \in N(b) \cap V'$. Moreover, if $d < b$ then $ad \notin E$. Now $cd \in E$ follows from $a < b < \{c, d\}$, when $b < d$, or from $a < d < b < c$ and $ad \notin E$, when $b > d$. □

In what follows we will need the following definition.

Definition 6. *A pseudo-sun* S_k *of size* k $(k \geq 3)$ *is a graph whose vertex set can be partitioned into two sets,* $U = \{u_1, \ldots, u_k\}$, $W = \{w_1, \ldots, w_k\}$, *so that* U *forms a cycle* $C_k = (u_1, \ldots, u_k, u_1)$, *the graph induced by* W *has no connected components with only two vertices, and* w_i *is adjacent to* u_j *if and only if* $i = j$ *or* $i = j + 1$ (*mod* k). *A sun is a pseudo-sun in which* U *is a clique (we call it inner clique) and* W *is an independent set.*

A pseudo–sun S_3 (hexahedron) and suns S_3 and S_4 are presented in Figure 3.

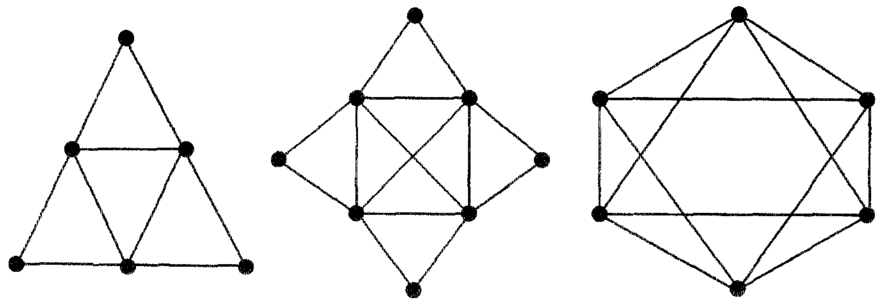

FIGURE 3. Some forbidden subgraphs.

In [5, 8] it was proven that a graph G is strongly chordal if and only if it is chordal and does not contain any sun S_k as an induced subgraph. A graph is called *weakly triangulated* [11] if it has no hole or antihole as an induced subgraph.

Corollary 2. *Every greedy matchable graph is a weakly triangulated graph that does not contain pseudo–suns and graphs from Figure 4 as induced subgraphs.*

Proof. Straightforward verification shows that no edge of hole or any graph from Figure 4 is simplicial. By Lemma 1, these graphs cannot be induced subgraphs of a greedy matchable graph. We will see also that no antihole and no pseudo–sun has a simplicial edge.

Consider an antihole $\bar{C}_k = (v_1, \ldots, v_k, v_1)$ with the edge set E and an arbitrary edge $v_i v_j$, $i \neq j \pm 1 \pmod{k}$, of it. Vertices v_i and v_j divide the cycle C_k into two induced paths. Let $v_l v_{l+1}$ be a middle edge of a longest path, and assume that v_l is closer than v_{l+1} to v_j in that path. Since $k \geq 5$ in $\bar{C}_k$ we have $v_l v_i, v_i v_j, v_j v_{l+1} \in E$ but $v_l v_{l+1} \notin E$, that is the edge $v_i v_j$ is not simplicial.

Now let S_k be a pseudo–sun with cycle $C_k = (u_1, \ldots, u_k, u_1)$ and vertices $\{w_1, \ldots, w_k\}$ of W. Consider an edge $w_i w_j$ of S_k. By definition of pseudo–sun, vertex u_i cannot be adjacent to both w_{j-1} and w_{j+1}. Assume without loss of generality that $w_{j+1} u_i \notin E$. Then from $u_i \in N(w_i)$, $w_{j+1} \in N(w_j)$ and $w_{j+1} u_i \notin E$ it follows that the edge $w_i w_j$ is not simplicial. Now consider the edges $w_i u_i$ and $w_i u_{i-1}$ of S_k. Since $u_i w_{i+1}, u_{i-1} w_{i-1} \in E$ and $u_i w_{i-1}, u_{i-1} w_{i+1} \notin E$ both these edges cannot be simplicial. Finally consider a possible edge $u_i u_j$ of S_k. Since a subgraph of S_k induced by W has no connected components with only one edge, we can find a third vertex u_l in W adjacent to u_i or/and to u_j. Assume without loss of generality that $u_i u_l \in E$. As $l \neq j$, vertex u_l cannot be adjacent to both w_{j+1} and w_j. Let $w_j u_l \notin E$. Then again from $u_l \in N(u_i)$, $w_j \in N(u_j)$ and $w_j u_l \notin E$ we get that the edge $u_i u_j$ is not simplicial. This completes the proof. □

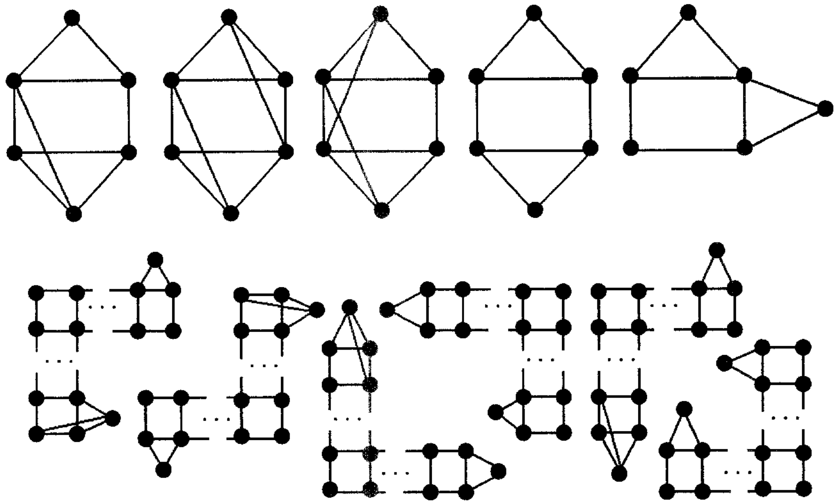

FIGURE 4. More forbidden subgraphs.

It seems, simplicial edges play an important role in the structure of greedy matchable graphs. We could not find any minimal forbidden subgraph of greedy matchable graphs which has a simplicial edge. Motivating by this we formulate the following conjecture.

Conjecture. *If every nontrivial induced subgraph of a graph G has a simplicial edge, then G is a greedy matchable graph.*

Below we will show that in two particular cases this conjecture is true.

Definition 7. *An ordering σ of the vertex set of a graph G is a lexicographic ordering if the following property holds.*

$(P\star)$ *For every three vertices a, b, c with $a < b$, $ac \in E$ and $bc \notin E$ there exists a vertex d with $d > c$, $db \in E$ and $da \notin E$.*

Lemma 2. *Any graph has a lexicographic ordering. Moreover, a lexicographic ordering of a graph $G = (V, E)$ can be produced in $O(|V||E|)$ time.*

Proof is omitted. In our proof we show that, for a given graph $G = (V, E)$, a graph theoretic variant of the algorithm, proposed by HOFFMAN, KOLEN and SAKAROVITCH in [12] for doubly lexical ordering of (0, 1)-matrices, gives a lexicographic ordering of G in $O(|V||E|)$ time.

A house is an induced C_4 with one additional vertex adjacent to exactly two adjacent vertices of C_4. A tent is an induced C_4 with an additional vertex adjacent to exactly three vertices of C_4 (see Figure 6).

Theorem 3. *Let G contain no induced subgraphs isomorphic to tent, house, sun, hexahedron or hole. Then every lexicographic ordering of G is a strong ordering.*

Proof is omitted.

Since suns, holes and hexahedron do not have any simplicial edges, house has only one and tent has two adjacent simplicial edges, we have

Theorem 4. *If every induced subgraph of a graph G, enjoying at least two nonadjacent edges, has two nonadjacent simplicial edges, then G is a greedy matchable graph and a greedy matching ordering of G can be found in $O(|V||E|)$ time.*

Strongly chordal graphs can be characterized also by another elimination scheme [8]. A vertex v of a graph G is called *simple* if for all $x, y \in N(v)$, $N[x] \subseteq N[y]$ or $N[y] \subseteq N[x]$ holds, i.e. $\{N[x] : x \in N[v]\}$ is linearly ordered by inclusion. A *simple elimination ordering* of a graph G is an ordering σ such that vertex v is simple in $G_{\geq v}$ for all $v \in V$. In [8] it is shown that G is strongly chordal if and only if G admits a simple elimination ordering. Note that a strong simplicial ordering is a simple elimination ordering σ such that for all $v \in V$ in $G_{\geq v}$ we have $N[x] \subseteq N[y]$ whenever $x < y$ in σ and $x, y \in N(v)$. Here we define a new ordering of the vertex set of a graph which has similar relation to strong orderings.

Definition 8. *A vertex v of a graph G is quasi–simple if for all $x, y \in N(v)$ the following holds:*
1. if $x, y \in E$ then $N[x] \subseteq N[y]$ or $N[y] \subseteq N[x]$,
2. if $x, y \notin E$ then $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$.
A quasi–simple elimination ordering of a graph G is an ordering σ such that vertex v is quasi–simple in $G_{\geq v}$ for all $v \in V$.

We will need also the following notion of a *simplicial-edge-without-vertex elimination ordering*. It generalizes the known notion of a *bisimplicial-edge-without-vertex elimination ordering* (see [3] and [16]) and refers to an edge elimination ordering such that no vertices are deleted in the process.

Definition 9. *Let $(e_1, \ldots, e_m)$ be an ordering of the edges of $G = (V, E)$ and $G_i = (V, E_i)$ be a subgraph of G with vertex set V and edge set $E_i = \{e_j : j \geq i\}$. The ordering $(e_1, \ldots, e_m)$ is a simplicial-edge-without-vertex elimination ordering for G if each edge e_i is simplicial in G_i.*

It is known from [3] (see also [16]) that a graph G is chordal bipartite if and only if G has a bisimplicial-edge-without-vertex elimination ordering.

The next result shows that both dismantling schemes defined above characterize the graphs with strong orderings.

Theorem 5. *The following three conditions are equivalent for a graph G.*

(1) *G admits a strong ordering.*
(2) *G admits a quasi–simple elimination ordering.*
(3) *G admits a simplicial-edge-without-vertex elimination ordering.*

Proof is omitted. Our proof is constructive and leads to $O(|V|^3)$ time algorithm for computing a strong ordering of G.

Theorem 6. *Every graph G that has a simplicial-edge-without-vertex elimination ordering is a greedy matchable graph and a greedy matching ordering of G can be found in* $O(|V|^3)$ *time.*

A greedy matchable graph that does not have any simplicial-edge-without-vertex elimination ordering is given in Figure 5. It has only one simplicial edge. After removing this edge we get a sun S_3 which is not a greedy matchable graph by Corollary 2.

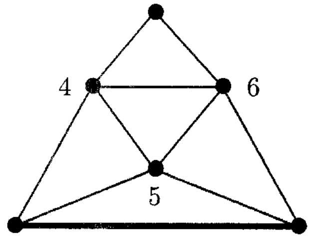

FIGURE 5. A greedy matchable graph that does not have any simplicial-edge-without-vertex elimination ordering.

4 More Subclasses

Here we describe two subclasses of greedy matchable graphs where greedy matching orderings can be found more efficiently. One of these subclasses contains both strongly chordal and chordal bipartite graphs. It is well-known [18, 23] that for these graphs any lexical ordering of LUBIW [18], defined below, is a strong ordering and hence strong orderings for these graphs can be produced in $O(min\{|V|^2, |E|\log|V|\})$ time [19, 23]. Note that strongly chordal graphs are exactly the class of chordal greedy matchable graphs, while chordal bipartite graphs are exactly bipartite greedy matchable graphs.

In what follows we will use properties :

$(P0)$ If $a < b$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $d \in N(b) \cup \{b\}$ and $da \notin E$. ($d = b$ is allowed.)

$(P1)$ If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $db \in E$!and $da \notin E$.

Evidently, $(P1)$ is a relaxation of $(P0)$ and $(P0)$ is a relaxation of $(P\star)$. It is well–known that any ordering of a graph G produced by Lexicographic Breadth–First–Search (LexBFS [22]) has property $(P1)$ (cf. [9]). Moreover, any ordering with property $(P1)$ can be produced by LexBFS as shown in [4]. Recall that LexBFS orders vertices of a graph by assigning numbers from $n = |V|$ to 1 in the following way: assign the number k to a vertex v (as yet unnumbered) which has lexically largest vector $(s_i : i = n, n-1, \ldots, k+1)$, where $s_i = 1$ if v is adjacent to the vertex numbered i, and $s_i = 0$ otherwise.

Definition 10. *An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the vertex set of a graph G is called a LexBFS–ordering if it satisfies the property $(P1)$. If σ satisfies the property $(P0)$ then it is called a lexical ordering of G.*

Note that for a given graph G, a LexBFS–ordering of G can be generated by LexBFS in linear time [9] while to date the fastest method – doubly lexical ordering of the (closed) neighborhood matrix of G [18] – producing a lexical ordering of G takes $O(|E|\log|V|)$ [19] or $O(|V|^2)$ [23] time. Notice that every lexical ordering of a graph is a LexBFS–ordering but not conversely.

Simplicial vertex can be defined also as a vertex which is not midpoint of an induced P_3. In [15] this notion was relaxed: A vertex is *semi–simplicial* if it is not a midpoint of an induced P_4. An ordering σ is a *semi–simplicial ordering* if vertex v is semi–simplicial in $G_{\geq v}$ for all $v \in V$. In [15] (see also [7]) the authors characterized the graphs for which every LexBFS–ordering is a semi–simplicial ordering as the HHD–free graphs, i.e. the graphs which contain no house, hole and domino as an induced subgraph (cf. Figure 6). We will use this fact in what follows.

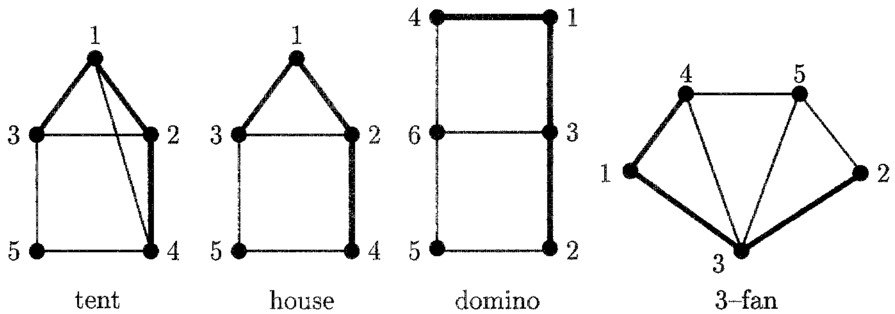

FIGURE 6. Orderings produced by LexBFS that are not admissible.

Lemma 3. *Let G contain no induced subgraphs isomorphic to tent, house, domino, hexahedron or hole, and let σ be a LexBFS–ordering (or a lexical ordering) of G. For every four vertices a, b, c, d of G such that $ab, ac, bd \in E$ and $\{a, b\} < \{c, d\}$, we have $cd \in E$.*

Proof. We prove the assertion for LexBFS–ordering only. This is enough because every lexical ordering of a graph G is a LexBFS–ordering of G. Assume that $cd \notin E$ for some vertices a, b, c, d with $ab, ac, bd \in E$ and $\{a, b\} < \{c, d\}$. We may suppose that $a < b$. Then $cb \in E$ or $ad \in E$ since a is a semi–simplicial vertex in $G_{\geq a}$.

If $cb \notin E$ then $ad \in E$, and applying $(P1)$ to $a < b < c$ yields a vertex $t > c$ adjacent to b but not to a. From semi–simpliciality of a we have $ct \in E$. But now the vertices a, b, c, d, t induce a house or a tent, a contradiction. Therefore, c and b are adjacent.

If $d < c$ then $(P1)$ applied to $b < d < c$ gives a vertex $t > c$ adjacent to d but not to b. Since the vertex b is minimal in the path formed by c, b, d, t it cannot be a middle vertex of an induced P_4. Hence, tc is an edge of G. To avoid a house and a tent we must have $at, ad \in E$. Now we can apply $(P1)$ to $a < d < c$ yielding a vertex $s > c$ adjacent to d but not to a. Note that $s \neq t$. Again, using semi–simpliciality of b in $G_{\geq b}$, we obtain $sc \in E$ or $sb \in E$. If $sc \in E$ then $sb \in E$, otherwise we get a tent. So, in every case the vertices s and b are adjacent. To avoid a house and a possible tent formed by c, b, d, t, s we must have both edges sc and st. Now the vertices c, b, d, t, s and a induce a hexahedron, a contradiction.

Thus, $d > c$. Applying $(P1)$ to $b < c < d$ we will get a vertex $t > d$ adjacent to c and not to b. From semi–simpliciality of b in $G_{\geq b}$ the vertices t and d must be adjacent. To avoid a house and a tent we must have $at, ad \in E$. Now we apply $(P1)$ to $a < b < t$ and find a vertex $s > t$ adjacent to b and not to a. From semi–simpliciality of a in $G_{\geq a}$, we have $ts \in E$. Furthermore, $cs \in E$ and $sd \in E$, otherwise we obtain a tent induced by $\{c, a, b, s, t\}$ or by $\{d, a, b, s, t\}$. But then all vertices together induce a hexahedron, a contradiction.

This settles the proof. □

The following theorem gives a characterization of those greedy matchable graphs for which every LexBFS–ordering is a greedy matching ordering.

Theorem 7. *Any LexBFS–ordering of a graph G is a greedy matching ordering if and only if G does not contain an induced subgraph isomorphic to tent, house, domino, 3–fan, hexahedron or hole.*

Proof. Suppose that G has no induced subgraphs isomorphic to tent, house, domino, 3–fan, hexahedron or hole, but however σ produced by LexBFS is not a greedy matching ordering. Then we can find four vertices a, b, c, d in G such that $ab, ac, bd \in E$, $a < d < b < c$, $ad \notin E$ and $cd \notin E$. Note that the case when $\{a, b\} < \{c, d\}$ is handled by Lemma 3. Again from semi–simpliciality of a in $G_{\geq a}$ we conclude that $cb \in E$. Property $(P1)$ applied to $a < d < c$ gives a vertex $t > c$ adjacent to d but not to a. Since the vertex d is minimal in the path formed by c, b, d, t this path cannot be induced, i.e. $tb \in E$ or $tc \in E$ holds in G. If $tb \notin E$ then $tc \in E$ and we obtain a house. Therefore, tb must be an edge, and the vertices a, b, c, t fulfill conditions $ab, ac, bt \in E$ and $\{a, b\} < \{c, t\}$. By Lemma 3, $ct \in E$. Hence, we have constructed in G an induced 3–fan that is impossible.

By Corollary 2, for the converse we need to show only that the graphs from Figure 6 must be forbidden. Assume that G contains a house or a tent as an induced subgraph. We start LexBFS with the vertex labeled by 5 in Figure 6 yielding number n. Let k, l, s and t be the numbers of vertices 4, 3, 2 and 1, respectively. Since vertices 2 and 1 are not adjacent to 5 but 4 and 3 are adjacent, we get $\{t, s\} < \{k, l\}$. From $kl \notin E$ we conclude that the obtained LexBFS–ordering is not a greedy matching ordering. Now assume that G contains a 3–fan. We start LexBFS with the vertex labeled by 5 yielding number n. Now we may number vertex 4 by $n - 1$ and vertex 3 by $n - 2$. Let k and l be the numbers of vertices 2 and 1, respectively. By the rules of LexBFS we get $k > l$. Since $l < k < n-2 < n-1$, $kl \notin E$ and the vertices $n-1$ and k are not adjacent, this LexBFS–ordering is not a greedy matching ordering. Finally assume that G contains a domino. We start LexBFS with the vertex labeled by 6 yielding number n. Then we may number vertex 5 by $n - 1$. Let k, l, s and t be the numbers of vertices 4, 3, 2 and 1, respectively. By the rules of LexBFS we get $t < s < \{k, l\}$. Suppose that we cannot number vertex k before vertex l in this LexBFS–ordering. Then $l > k$ and there exists a vertex v such that $v > l$, $vl \in E$ and $vk \notin E$. Since $v > l$ by the rules of LexBFS we must have $vn \in E$. But now the vertices n, k, l, t and v induce either a hose or a tent. This contradicts to the proof above. So, we may number the vertex k before l and get $t < s < l < k$. Since $st, sk \notin E$ again the obtained LexBFS–ordering is not a greedy matching ordering. □

The graphs that do not contain house, hole, domino and 3–fan as induced subgraphs are known as *distance–hereditary graphs* [13]. *Ptolemaic graphs* are exactly the chordal distance–hereditary graphs [14] while *bipartite (6,2)-chordal graphs* are the bipartite distance–hereditary graphs [2].

Corollary 3. *Let G be a ptolemaic graph, or a bipartite (6,2)-chordal graph, or a distance–hereditary graph without induced subgraphs isomorphic to tent and hexahedron. Then a maximum matching of G can be computed in linear time.*

Denote by S_k^- the graph isomorphic to sun S_k without one vertex from the independent set W (see Definition 6).

Theorem 8. *Any lexical ordering of a graph G is a greedy matching ordering if G does not contain an induced subgraph isomorphic to tent, house, domino, sun, hexahedron or hole.*

Proof. By contradiction, as in the proof of Theorem 7 we will find four vertices x_1, y_1, y_2, x_2 in G such that $x_1y_1, x_1y_2, y_1x_2 \in E$, $x_1 < x_2 < y_1 < y_2$, $x_1x_2 \notin E$ and $y_2x_2 \notin E$. We may choose these vertices x_1, y_1, y_2, x_2 with maximal sum $\Sigma = x_1 + y_1 + y_2 + x_2$ of their numbers in σ.

Repeating the arguments of the proof of Theorem 7 (we need only to replace $(P1)$ by $(P0)$ and a, b, c, d, t by x_1, y_1, y_2, x_2, y_3, respectively), we can construct in G an induced 3–fan (S_3^-) formed by x_1, y_1, y_2, x_2, y_3 with inner clique $\{y_1, y_2, y_3\}$, independent set $\{x_1, x_2\}$ and $x_1 < x_2 < y_1 < y_2 < y_3$. We may choose the vertex y_3 with $y_3 > y_2$, $y_3x_2 \in E$ and $y_3x_1 \notin E$ rightmost in σ.

In what follows we show how to extend this S_3^- to an induced S_4^-. Since $y_1 < y_2$ and $x_2y_1 \in E$ but $x_2y_2 \notin E$ we can apply $(P0)$ and get a vertex $x_3 > x_2$ adjacent to y_2 and not to y_1. Note that $x_3 \neq y_2$. The vertices x_3 and y_3 are not adjacent, for otherwise we would have a 3–sun or a house or a tent as an induced subgraph (it depends on adjacency between x_3 and x_1, x_2).

We claim that $x_3 < y_1$. Indeed, if $x_3 > y_1$ then either $y_1 < x_3 < y_2 < y_3$ or $y_1 < y_2 < \{x_3, y_3\}$. Since $y_1y_2, y_1y_3, y_2x_3 \in E$ and $x_3y_1 \notin E$ we obtain $x_3y_3 \in E$ from maximality of Σ (note that $x_1 + y_1 + y_2 + x_2 < y_1 + y_2 + y_3 + x_3$) or by Lemma 3, respectively. But this is impossible. Hence, $x_3 < y_1$.

Now we apply $(P0)$ to $x_2 < x_3 < y_3$ yielding a vertex $y_4 > y_3$ adjacent to x_3 and not to x_2. We choose the vertex y_4 rightmost in σ. Since $\{x_3, y_2\} < \{y_3, y_4\}$, by Lemma 3, y_3 and y_4 are adjacent. Moreover, to avoid a house or a tent formed by $\{y_1, y_2, y_3, y_4, x_3\}$ or a 3–sun formed by $\{y_1, y_2, y_3, y_4, x_1, x_2\}$, we must have $y_2y_4, y_1y_4 \in E$. The vertices y_4 and x_1 cannot be adjacent. For otherwise, from $x_1 < x_2 < y_1 < y_4$, $x_1x_2 \notin E$ and maximality of Σ we would get $y_4x_2 \in E$ that is impossible. Furthermore, $x_3x_1, x_3x_2 \notin E$ because induced house and tent are forbidden subgraphs for G.

Thus, we have constructed in G an induced S_4^- with inner clique $\{y_1, y_2, y_3, y_4\}$, independent set $\{x_1, x_2, x_3\}$ and $x_1 < x_2 < x_3 < y_1 < y_2 < y_3 < y_4$.

Next we show how to extend induced S_{k-1}^- to an induced S_k^- for an arbitrary $k \geq 5$ (this is done in the full version). Since we deal with finite graphs, a contradiction arises. □

Notice that the class of graphs described in Theorem 8 does not contain all chordal bipartite graphs. To get a greedy matching ordering for such a graph we have to apply the doubly lexical ordering method to the bipartite adjacency matrix instead to the neighborhood matrix, as in the case of general graphs[18, 16].

As far as the orderings of the first three graphs presented in Figure 6 are lexical but not admissible, we can conclude

For every induced subgraph F of a graph G any lexical ordering is a greedy matching ordering if and only if G does not contain an induced subgraph isomorphic to tent, house, domino, sun, hexahedron or hole.

Summarizing we have

Corollary 4. *For a graph G that does not contain an induced subgraph isomorphic to tent, house, domino, sun, hexahedron or hole, a maximum matching can be computed in time $O(min\{|V|^2, |E|\log|V|\})$. If G is given together with a lexical ordering then a maximum matching of G can be computed in linear time.*

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