LexBFS–Orderings and Powers of Graphs*

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Abstract. For an undirected graph G the k-th power G^k of G is the graph with the same vertex set as G where two vertices are adjacent iff their distance is at most k in G. In this paper we consider LexBFS–orderings of chordal, distance–hereditary and HHD–free graphs (the graphs where each cycle of length at least five has two chords) with respect to their powers. We show that any LexBFS–ordering of a chordal graph is a common perfect elimination ordering of all odd powers of this graph, and any LexBFS–ordering of a distance–hereditary graph is a common perfect elimination ordering of all its even powers. It is well–known that any LexBFS–ordering of a HHD–free graph is a so–called semi–simplicial ordering. We show, that any LexBFS–ordering of a HHD–free graph is a common semi–simplicial ordering of all its odd powers. Moreover we characterize those chordal, distance–hereditary and HHD–free graphs by forbidden isometric subgraphs for which any LexBFS–ordering of the graph is a common perfect elimination ordering of all its nontrivial powers. As an application we get a linear time approximation of the diameter for weak bipolarizable graphs, a subclass of HHD–free graphs containing all chordal graphs, and an algorithm which computes the diameter and a diametral pair of vertices of a distance–hereditary graph in linear time.

1 Introduction

Powers of graphs play an important role for solving certain problems related to distances in graphs : p–center and q–dispersion (cf. [7, 3]), k–domination and k–stability (cf. [8, 3]), diameter (cf. [13]), k–colouring (cf. [26, 20]) and approximation of bandwidth (cf. [27]). For instance, consider the k–colouring problem. The vertices of a graph have to be coloured by a minimal number of colours such that no two vertices at distance at most k have the same colour. Obviously, k–colouring a graph is equivalent to colour (in the classical sense)

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its k-th power. It is well–known that the colouring problem is $\mathbb{NP}$–complete in general. On the other hand, there are a lot of special graph classes with certain structural properties for which the colouring problem is efficiently solvable. One of the most popular class is the one of chordal graphs. Here we have a linear time colouring algorithm by stepping through a certain dismantling scheme — the so–called perfect elimination ordering — of the graph. So it is quite natural to consider graph classes for which certain powers are chordal.

In the last years some papers investigating powers of chordal graphs were published. One of the first results in this field is due to DUCHET ([18]) : If G^k is chordal then G^{k+2} is so. In particular, odd powers of chordal graphs are chordal, whereas even powers of chordal graphs are in general not chordal. Chordal graphs with chordal square were characterized by forbidden configurations in [28].

It is well–known that any chordal graph has a perfect elimination ordering which can be computed in linear time by Lexicographic Breadth–First–Search (LexBFS, [32]) or Maximum Cardinality Search (MCS, [33]). Thus each chordal power of an arbitrary graph has a perfect elimination ordering. A natural question is whether there is a common perfect elimination ordering of all (or some) chordal powers of a given graph. The first result in this direction using minimal separators is given in [17] : If both G and G^2 are chordal then there is a common perfect elimination ordering of these graphs (see also [4]). The existence of a common perfect elimination ordering of all chordal powers of an arbitrary given graph was proved in [3]. Such a common ordering can be computed in time $O(|V||E|)$ using a generalized version of Maximum Cardinality Search which simultaneously uses chordality of these powers.

Here we consider the question whether LexBFS, working only on an initial graph G, produces a common perfect elimination ordering of chordal powers of G. Hereby we consider chordal, distance–hereditary and HHD–free graphs as initial graphs. Recall, that in chordal graphs every cycle of length at least four has a chord and in distance–hereditary graphs each cycle of length at least five has two crossing chords. HHD–free graphs can be defined as the graphs in which every cycle of length at least five has two chords. Analogously to chordal graphs, HHD–free graphs can be dismantled via a so–called semi–simplicial ordering which can be produced in linear time by LexBFS (cf. [25]). Since a semi–simplicial ordering in reverse order is a perfect ordering (in sense of CHVATAL), HHD–free graphs are perfectly orderable, and hence they can be coloured in linear time (cf. [10]).

2 Preliminaries

Throughout this paper all graphs $G = (V, E)$ are finite, undirected, simple (i.e. loop–free and without multiple edges) and connected.

A *path* is a sequence of vertices $v_0, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 0, \ldots, k-1$; its *length* is k. As usual, an induced path of k vertices is denoted by P_k. A graph G is *connected* iff for any pair of vertices of G there is a path in G joining both vertices.

The *distance* $d_G(u,v)$ of vertices u,v is the minimal length of any path connecting these vertices. Obviously, d_G is a metric on G. If no confusion can arise we will omit the index G. An induced subgraph H of G is an *isometric* subgraph of G iff the distances within H are the same as in G, i.e.

$$\forall x,y \in V(H) \,:\, d_H(x,y) = d_G(x,y).$$

The *k–th neighbourhood* $N^k(v)$ of a vertex v of G is the set of all vertices of distance k to v, i.e.

$$N^k(v) := \{u \in V : d_G(u,v) = k\},$$

whereas the *disk* of radius k centered at v is the set of all vertices of distance at most k to v :

$$D_G(v,k) := \{u \in V : d_G(u,v) \leq k\} \;=\; \bigcup_{i=0}^{k} N^i(v).$$

For convenience we will write $N(v)$ instead of $N^1(v)$. Again, if no confusion can arise we will omit the index G. The *k-th power* G^k of G is the graph with the same vertex set V where two vertices are adjacent iff their distance is at most k. If $k \geq 2$ then G^k is called *nontrivial power.*

The *eccentricity* $e(v)$ of a vertex $v \in V$ is the maximum over $d(v,x)$, $x \in V$. The minimum over the eccentricities of all vertices of G is the *radius* $rad(G)$ of G, whereas the maximum is the *diameter* $diam(G)$ of G. A pair x,y of vertices of G is called *diametral* iff $d(x,y) = diam(G)$.

Next we recall the definition and some characterizations of chordal graphs. An *induced cycle* is a sequence of vertices $v_0,\ldots,v_k$ such that $v_0 = v_k$ and $v_iv_j \in E$ iff $|i-j| = 1$ (modulo k). The *length* $|C|$ of a cycle C is its number of vertices. A graph G is *chordal* iff any induced cycle of G is of length at most three. One of the first results on chordal graphs is the characterization via dismantling schemes. A vertex v of G is called *simplicial* iff $D(v,1)$ induces a complete subgraph of G. A *perfect elimination ordering* is an ordering of G such that v_i is simplicial in $G_i := G(\{v_i,\ldots,v_n\})$ for each $i = 1,\ldots,n$. It is well–known that a graph is chordal if and only if it has a perfect elimination ordering (cf. [21]). Moreover, computing a perfect elimination ordering of a chordal graph can be done in linear time by Lexicographic Breadth–First–Search (LexBFS, [21]). To make the paper self–contained we present the rules of this algorithm.

Let $s_1 = (a_1,\ldots,a_k)$ and $s_2 = (b_1,\ldots,b_l)$ be vectors of positive integers. Then s_1 is *lexicographically smaller* than s_2 ($s_1 < s_2$) iff

1. there is an index $i \leq \min\{k,l\}$ such that $a_i < b_i$ and $a_j = b_j$ for all $j = 1,\ldots,i-1$, or
2. $k < l$ and $a_i = b_i$ for all $i = 1,\ldots,k$.

If $s = (a_1,\ldots,a_k)$ is a vector and a is some positive integer then $s + a$ denotes the vector $(a_1,\ldots,a_k,a)$.

procedure LexBFS
Input : A graph $G = (V, E)$.
Output : A LexBFS–ordering $\sigma = (v_1, \ldots, v_n)$ of V.

```
begin forall v ∈ V do l(v) := ();
      for n := |V| downto 1 do
          choose a vertex v ∈ V with lexicographically maximal label l(v);
          define σ(n) := v;
          forall u ∈ V ∩ N(v) do l(u) := l(u) + n;
          V := V \ {v};
      endfor;
end.
```

In the sequel we will write $x < y$ whenever in a given ordering of the vertex set of a graph G vertex x has a smaller number than vertex y. Moreover, $x < \{y_1, \ldots, y_k\}$ is an abbreviation for $x < y_i$, $i = 1, \ldots, k$.

In what follows we will often use the following property (cf. [25]) :

$(P1)$ If $a < b < c$ and $ac \in E$ and $bc \notin E$ then there exists a vertex d such that $c < d$, $db \in E$ and $da \notin E$.

Lemma 1. (1) *Any LexBFS–ordering has property* $(P1)$.
(2) *Any ordering fulfilling* $(P1)$ *can be generated by LexBFS.*

Proof. (1) We refer to the well–known proof in [21].
(2) Let $\sigma = (v_1, \ldots, v_n)$ be an ordering fulfilling $(P1)$ and suppose that $(v_{i+1}, \ldots, v_n)$, $i \leq n-1$, can be produced by LexBFS but not $(v_i, \ldots, v_n)$, i.e. v_i cannot be chosen via LexBFS. Let u be the vertex chosen next by LexBFS. Then there must be a vertex $w > v_i$ adjacent to u but not to v_i. We can choose w rightmost in σ. Thus in σ we have $u < v_i < w$, $uw \in E$ and $wv_i \notin E$. Now $(P1)$ implies the existence of a vertex $z > w$ adjacent to v_i but not to u. Since w is chosen rightmost all vertices with a greater number than w which are adjacent to u are adjacent to v_i too. Hence the LexBFS–label of v_i is greater than that of u, a contradiction. □

3 Chordal Graphs

A set $S \subseteq V$ is *m–convex* (monophonically convex) iff for all pairs of vertices x, y of S each vertex of any induced path connecting x and y is contained in S too.

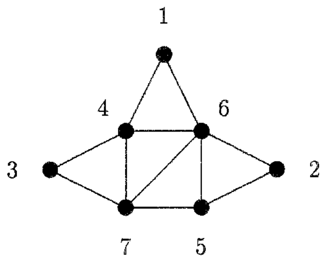
Lemma 2 [19]. *If G is a chordal graph and $(v_1, \ldots, v_n)$ is a perfect elimination ordering of G then $V(G_i)$ is m–convex in G and, in particular, G_i is an isometric subgraph of G, for every $i = 1, \ldots, n$.*

Using property $(P1)$, m–convexity and isometricity of G_i in G we can prove

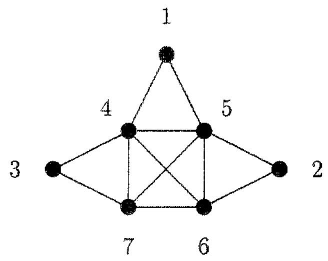
Theorem 3. *For a chordal graph G every LexBFS–ordering of G is a perfect elimination ordering of each odd power G^{2k+1} of G.*

Since we do not use chordality of odd powers in the proof of the above theorem we reproved that odd powers of chordal graphs are again chordal.

Theorem 4. *If G is a chordal graph which does not contain the graphs of Figure 1 as isometric subgraphs then every LexBFS–ordering of G is a perfect elimination ordering of each even power G^{2k}, $k \geq 1$, of G.*



F_1



F_2

Fig. 1. Chordal graphs labeled by a LexBFS–ordering such that vertex 1 is not simplicial in G^2.

Corollary 5. *If G is chordal and does not contain the graphs of Figure 1 as isometric subgraphs then all powers of G are chordal.*

Ptolemaic graphs (cf. [9, 24]) are the graphs fulfilling the ptolemaic inequality, i.e. for any four vertices u, v, w, x it holds

$$d(u,v)d(w,x) \leq d(u,w)d(v,x) + d(u,x)d(v,w).$$

In [24] it was shown that the ptolemaic graphs are exactly the chordal graphs without a 3–fan (cf. Figure 4), i.e. the distance–hereditary chordal graphs (cf. [2]). For the well–known class of interval graphs we refer to [21].

Corollary 6. *If G is a ptolemaic or interval graph then any LexBFS–ordering of G is a common perfect elimination ordering of all powers of G.*

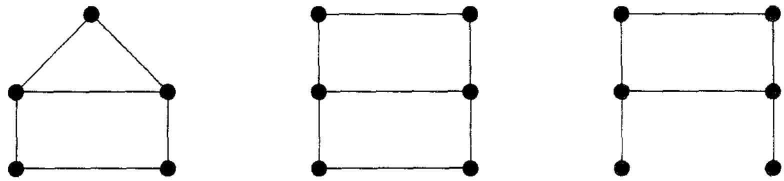
Corollary 7. *If G is a ptolemaic or interval graph and v is the first vertex of a LexBFS–ordering of G, then $e(v) = diam(G)$.*

Proof. Let σ be a LexBFS–ordering of G, v be the first vertex of σ and k its eccentricity. By Corollary 6 σ is a perfect elimination ordering of the power G^k of G. In particular, v is simplicial in G^k. Thus G^k is complete. □

Hence the diameter and a diametral pair of vertices of a ptolemaic or interval graph can be computed in linear time by only using a LexBFS–ordering.

4 HHD–free Graphs

Note that a vertex is simplicial if and only if it is not midpoint of a P_3. In [25] this notion was relaxed : A vertex is *semi–simplicial* iff it is not a midpoint of a P_4. An ordering $(v_1, \dots, v_n)$ is a *semi–simplicial ordering* iff v_i is semi–simplicial in G_i for all $i = 1, \dots, n$. In [25] the authors characterized the graphs for which every LexBFS–ordering is a semi–simplicial ordering as the HHD–free graphs, i.e. the graphs which do not contain a house, hole or domino as induced subgraph (cf. Figure 2).



The house. The domino. The 'A'.

Fig. 2. The house, the domino and the 'A'.

If a HHD–free graph does not contain the 'A' of Figure 2 as induced subgraph then this graph is called *weak bipolarizable* (HHDA–free) [31].

In [16] we investigated powers of HHD–free graphs. We proved that odd powers of HHD–free graphs are again HHD–free. Furthermore, an odd power G^{2k+1} of a HHD–free graph G is chordal if and only if G does not contain a $C_4^{(k)}$ as an isometric subgraph (cf. [1] and [5] for the role of $C_4^{(k)}$ in distance–hereditary graphs and hole–free graphs). Hereby, a $C_4^{(k)}$ is a graph induced by a C_4 with pendant paths of length k attached to the vertices of the C_4, see Figure 3.

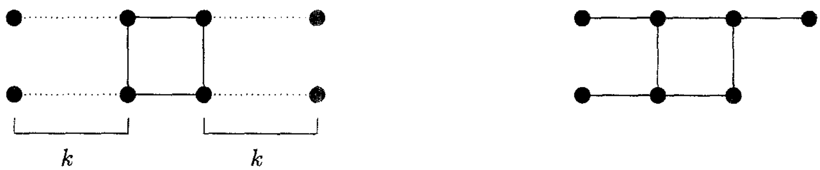


Fig. 3. A $C_4^{(k)}$ and the $C_4^{(1)}$ minus a pendant vertex.

As a relaxation of m–convexity in chordal graphs we introduced the notion of m^3–convexity in [15] : A subset $S \subseteq V$ is called *m^3–convex* iff for any pair of vertices x, y of S each induced path of length at least 3 connecting x and y is completely contained in S.

Lemma 8 [15]. *An ordering $(v_1, \ldots, v_n)$ of the vertices of a graph G is semi–simplicial if and only if $V(G_i)$ is m^3–convex in G for all $i = 1, \ldots, n$.*

The above lemma implies that the minimum (with respect to a semi–simplicial ordering) of an induced path of length at least three must be one of its endpoints.

The proofs of our results are based on nice properties of shortest paths in HHD–free graphs with respect to a given LexBFS–ordering.

Let $P = x_0 - \ldots - x_k$ be an induced path and σ be a LexBFS–ordering of the vertices of a HHD–free graph G. A vertex x_i, $1 \leq i \leq k-1$, is called *switching point* of P iff $x_{i-1} < x_i > x_{i+1}$ or $x_{i-1} > x_i < x_{i+1}$. The path P is *locally maximal* (with respect to σ) iff each vertex $y \in V \setminus V(P)$ which is adjacent to x_{i-1} and x_{i+1}, $1 \leq i \leq k-1$, is smaller than x_i, i.e. $y < x_i$. If P is not locally maximal then there must be a vertex x_i of P, $1 \leq i \leq k-1$, and a vertex $y \notin V(P)$ adjacent to x_{i-1} and x_{i+1} such that $x_i < y$.

Lemma 9. *Let $P = x_0 - \ldots - x_k$ be a shortest path, $k \geq 3$. Then*

1. *The number s of switching points of P is at most three.*
2. *The switching points of P induce a subpath of P.*
3. *If P is locally maximal then $s \leq 1$.*

Lemma 10. *Let $P = x_0 - \ldots - x_k$, $k \geq 3$, be a shortest path which is locally maximal. Furthermore let $x_0 < x_k$ and let x_i, $1 \leq i \leq k-1$, be the switching point of P. Then*

1. *$d(x_0, x_i) \geq d(x_i, x_k)$ and*
2. *if $d(x_0, x_i) = d(x_i, x_k)$, i.e. $k = 2i$, then $x_0 < x_k < \ldots < x_j < x_{k-j} < \ldots < x_{i-1} < x_{i+1} < x_i$.*

Using property $(P1)$, m^3–convexity and the above path properties we can show

Theorem 11. *Any LexBFS–ordering of a HHD–free graph G is a common semi–simplicial ordering of all odd powers of G.*

Theorem 12. *Any LexBFS–ordering of a HHD–free graph G is a common perfect elimination ordering of all nontrivial odd powers of G if and only if G does not contain a $C_4^{(1)}$ minus a pendant vertex (cf. Figure 3) as isometric subgraph.*

Corollary 13. *Any LexBFS–ordering of a weak bipolarizable graph is a common perfect elimination ordering of all its nontrivial odd powers.*

Corollary 14. *Let v be the first vertex of a LexBFS–ordering of a weak bipolarizable graph G. Then $diam(G) - 1 \leq e(v) \leq diam(G)$.*

Proof. First note that for $e(v) = 1$ there is nothing to show. If $e(v) = 2k + 1$, $k \geq 1$, then G^{2k+1} is complete and hence $diam(G) = e(v)$. For $e(v) = 2k$ the odd power G^{2k+1} is complete implying $diam(G) \leq 2k + 1 = e(v) + 1$. □

Theorem 15. *Any LexBFS–ordering of a HHD–free graph G is a common perfect elimination ordering of all even powers of G if and only if G does not contain one of the graphs of Figure 1 as isometric subgraph.*

Similar to Corollary 14 we can prove

Corollary 16. *Let v be the first vertex of a LexBFS–ordering of a HHD–free graph G which does not contain a graph of Figure 1 as isometric subgraph. Then* $diam(G) - 1 \leq e(v) \leq diam(G)$.

5 Distance–Hereditary Graphs

A graph G is *distance–hereditary* ([23]) iff each connected induced subgraph of G is isometric. Distance–hereditary graphs were extensively studied in [2], [22], [11], [1] and [29]. For proving our results we used the following property :

Theorem 17 (The four–point condition [2]). *Let G be a distance–hereditary graph. Then, for any four vertices u, v, w, x at least two of the distance sums*

$$d(u,v) + d(w,x), \quad d(u,w) + d(v,x), \quad d(u,x) + d(w,v)$$

are equal, and, if the two smaller sums are equal then the larger one exceeds this by at most two.

Furthermore, distance–hereditary graphs can be characterized by forbidden subgraphs ([2], [22]) : A graph is distance–hereditary if and only if it does not contain a hole, a house, a domino and a 3–fan as induced subgraph (see Figure 4).

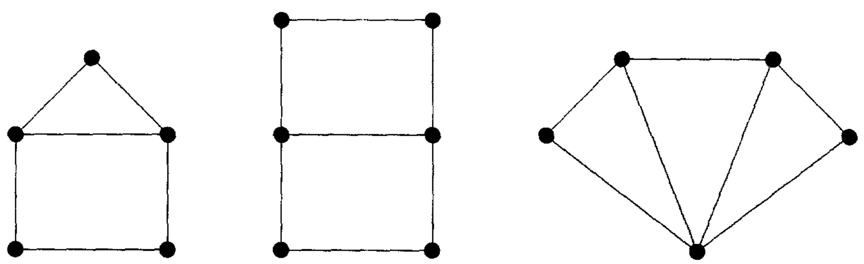


Fig. 4. A house, a domino and a 3–fan.

Thus distance–hereditary graphs are HHD–free, and each LexBFS–ordering of G is a semi–simplicial ordering of G.

Using the four–point condition, m^3–convexity and property $(P1)$ we can show

Theorem 18. *Each LexBFS–ordering σ of a distance–hereditary graph G is a perfect elimination ordering of each even power G^{2k}, $k \geq 1$.*

Thus we reproved that even powers of distance–hereditary graphs are chordal (cf. [1]). In [1] it was proved that all odd powers of a distance–hereditary graph are HHD–free. Moreover, an odd power G^{2k+1} is chordal if and only if G does not contain an induced subgraph isomorphic to the $C_4^{(k)}$, cf. Figure 3.

Theorem 19. *Any LexBFS–ordering σ of a given distance–hereditary graph G is a common perfect elimination ordering of all its nontrivial powers if and only if G does not contain a $C_4^{(1)}$ minus a pendant vertex (cf. Figure 3) as induced subgraph.*

Theorem 20. *Any LexBFS–ordering σ of a distance–hereditary graph G is a common semi–simplicial ordering of all its powers.*

Computing a diametral pair of vertices

In [12] a linear time algorithm for computing the diameter of a distance–hereditary graph was presented, but that approach is not usable for finding a diametral pair of vertices. As an application of the preceding results we present a simpler algorithm which computes both the diameter and a diametral pair of vertices of a distance–hereditary graph in linear time. This points out once more the importance of considering chordal powers of graphs and perfect elimination orderings of them.

Lemma 21. *Let v be the first vertex of a LexBFS–ordering of a distance–hereditary graph G. Then*

$$diam(G) - 1 \leq e(v) \leq diam(G).$$

Moreover, if $e(v)$ is even then $e(v) = diam(G)$.

Proof. If $e(v) = 2k$, $k \geq 1$, then G^{2k} is complete by Theorem 18, and thus $diam(G) = 2k$. If $e(v) = 2k + 1$, $k \geq 1$, then G^{2k+2} is complete by Theorem 18, and hence $2k + 1 \leq diam(G) \leq 2k + 2$. □

Corollary 22. *Let G be a distance–hereditary graph which does not contain a $C_4^{(1)}$ minus a pendant vertex (cf. Figure 3) as induced subgraph, and let v be the first vertex of a LexBFS–ordering of G. Then $e(v) = diam(G)$.*

Recall that the ptolemaic graphs are exactly the chordal distance–hereditary graphs. Thus they do not contain a $C_4^{(1)}$ minus a pendant vertex. Therefore, any LexBFS–ordering of a ptolemaic graph is a diametral ordering. In [30] such an ordering is used to check the Hamiltonicity of a ptolemaic graph in linear time.

For the sequel we may assume that G is not complete for otherwise there is nothing to do. In what follows we describe the steps of the algorithm.

At first we compute a LexBFS–ordering σ of a given distance–hereditary graph G. Let v be the first vertex of σ. If $e(v) = 2k$, $k \geq 1$, then, by Lemma 21, $e(v) = diam(G)$, and the vertices v and $w \in N^{e(v)}(v)$ form a diametral pair of G. So let $e(v) = 2k + 1$. Now we start LexBFS at vertex v yielding a LexBFS–ordering τ with first vertex u. If $e(u) = 2k + 2$ then, by Lemma 21, $diam(G) = 2k + 2$ and the vertices u and $w \in N^{e(u)}(u)$ form a diametral pair of G. Otherwise ($e(v) = e(u) = 2k + 1$) we choose a vertex z at distance k to u and at distance $k + 1$ to v.

Lemma 23. $k + 1 \leq e(z) \leq k + 2$.

Proof. Since $d(z, v) = k + 1$ we immediately have $e(z) \geq k + 1$. So let w be a vertex of V such that $d(z, w) \geq k + 2$. We obtain the following distance sums :

$$\begin{aligned} d(u,v) + d(z,w) &= 2k + 1 + d(z,w) \geq 3k + 3 \\ d(u,z) + d(v,w) &= k + d(v,w) \qquad\quad \leq 3k + 1 \\ d(u,w) + d(v,z) &= k + 1 + d(u,w) \quad \leq 3k + 2 \end{aligned}$$

Now the four–point condition gives

$$d(v, w) = 2k + 1, \quad d(u, w) = 2k, \quad \text{and} \quad d(z, w) = k + 2.$$

This settles the proof. □

For every vertex w of $V \setminus D(z, k)$ we store in *track*(w) the second edge of an arbitrary shortest path from z to w. Define $F := \{track(w) : w \in V \setminus D(z, k)\}$. We will say that two edges in a graph are *independent* iff the vertices of this edges induce a $2K_2$ in G.

Lemma 24. *$diam(G) = 2k+2$ if and only if the set F contains two independent edges.*

Proof. Let $diam(G) = 2k + 2$ and let x, y be vertices of G such that $d(x, y) = 2k + 2$. Since both u and v (as first vertices of LexBFS–orderings) are simplicial in G^{2k} we get

$$d(u, x) = d(u, y) = d(v, x) = d(v, y) = 2k + 1.$$

With $d(z, u) = k$ this implies $d(z, x) \geq k + 1$. So we obtain the following distance sums :

$$\begin{aligned} d(u,v) + d(z,x) &= 2k + 1 + d(z,x) \geq 3k + 2 \\ d(u,z) + d(v,x) &= k + 2k + 1 \qquad\quad = 3k + 1 \\ d(u,x) + d(v,z) &= 2k + 1 + k + 1 \quad = 3k + 2 \end{aligned}$$

Now the four–point condition gives $d(z, x) = k+1$. By symmetry, $d(z, y) = k+1$. Thus z lies on a shortest path joining x and y. Obviously, *track*(x) and *track*(y) are independent edges due to $d(x, y) = 2k + 2$ and $d(x, z) = d(y, z) = k + 1$.

Now let $s_1 s_2$ and $t_1 t_2$ be independent edges in F. Let $z - s_1 - s_2 - \ldots - w_1$ and $z - t_1 - t_2 - \ldots - w_2$ be shortest paths of length at least $k + 1$. We will prove

$d(w_1, w_2) = 2k + 2$. Since $s_2 - s_1 - z - t_1 - t_2$ is induced we get $d(s_2, t_2) = 4$. Using $k + 1 \leq e(z) \leq k + 2$ we obtain the following distance sums :

$$\begin{aligned}
d(w_1, z) + d(s_2, t_2) &= 4 + d(w_1, z) \in \{k+5, k+6\} \\
d(w_1, s_2) + d(z, t_2) &= 2 + d(w_1, s_2) \in \{k+1, k+2\} \\
d(w_1, t_2) + d(z, s_2) &= 2 + d(w_1, t_2)
\end{aligned}$$

Since the difference between the first and second distance sum is at least three the four–point condition implies that the larger two sums must be equal, i.e. the first and third one. So we get

$$k + 3 \leq d(w_1, t_2) \leq k + 4 \quad \text{and} \quad k + 3 \leq d(w_2, s_2) \leq k + 4$$

by symmetry. Together with $d(s_2, t_2) = 4$ this implies

$$\begin{aligned}
d(w_1, w_2) + d(s_2, t_2) &= 4 + d(w_1, w_2) \\
d(w_1, s_2) + d(w_2, t_2) &\in \{2k-2, 2k-1, 2k\} \\
d(w_1, t_2) + d(w_2, s_2) &\in \{2k+6, 2k+7, 2k+8\}
\end{aligned}$$

By the same argument as above the four–point condition implies that the first and the third distance sum must be equal, i.e. $d(w_1, w_2) \geq 2k + 2$. □

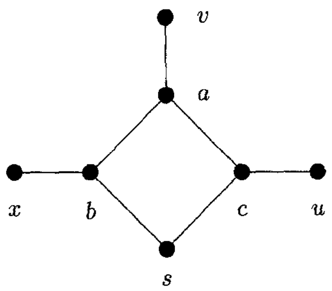
Therefore the following algorithm correctly computes the diameter and a diametral pair of a distance–hereditary graph :

Algorithm DHGDiam.

Input : A connected distance–hereditary graph G.
Output : $diam(G)$ and a diametral pair of vertices of G.

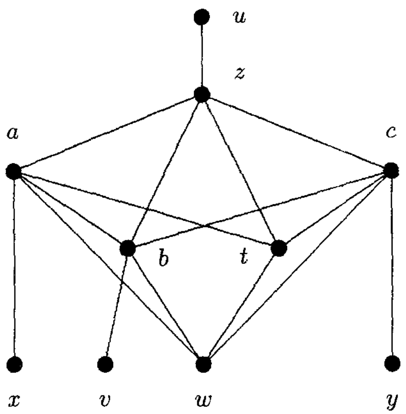
(1) **begin** σ :=LexBFS(G, s) for some $s \in V(G)$.
(2) Let v be the first vertex of σ.
(3) **if** $e(v)$ is even **then return**($e(v), (v, w)$) where $w \in N^{e(v)}(v)$.
(4) **else** τ :=LexBFS(G, v).
(5) Let u be the first vertex of τ.
(6) **if** $e(u) = e(v) + 1$ **then return**($e(u), (u, w)$) where $w \in N^{e(u)}(u)$.
(7) **else** Let $k \in \mathbb{N}$ such that $e(v) = e(u) = 2k + 1$.
(8) Choose a vertex z from $D(u, k) \cap D(v, k + 1)$.
(9) $F := \{track(w) : w \in V \setminus D(z, k)\}$.
(10) **if** F contains a pair e_1, e_2 of independent edges
(11) **then return**($2k + 2, (x, y)$)
where $x, y \in V$ such that $track(x) = e_1$ and $track(y) = e_2$.
(12) **else return**($2k + 1, (v, u)$)
(13) **end.**

Before going into the implementation details consider the examples of Figure 5. In the first one, a $C_4^{(1)}$ minus a pendant vertex, the algorithm correctly stops in step (6). In the second one both first vertices of both LexBFS–orderings have odd eccentricity. Thus we must compute the *track*–values and the set F.



$\sigma = (v, u, x, a, c, b, s)$
$\tau = (u, x, s, c, b, a, v)$

$e(v) = 3$
$diam(G) = e(u) = 4$



$\sigma = (v, x, y, w, b, a, c, t, z, u)$
$\tau = (u, x, y, t, z, a, c, w, b, v)$

$e(v) = e(u) = 3$
$diam(G) = d(x, y) = 4$

$F = \{xa, yc, vb, wb\}$
xa, yc independent

Fig. 5. Algorithm DHGDiam — Examples.

It remains to show that the above algorithm can be implemented to run in linear time. It is well-known that LexBFS and BFS run in linear time. So it is sufficient to consider steps (9) and (10).

Step (9). At first we build a BFS–tree rooted at z yielding the set of neighbourhoods $N^i(z)$, $i = 0, \ldots, e(z)$ of z. For any vertex $x \in V \setminus \{z\}$ let $f(x)$ denote the father of x in the BFS–tree.

We compute the *track*–values levelwise : For all vertices w in $N^2(z)$ define $track(w) := wy$ where $y = f(w)$. Recursively we compute $track(w) := track(f(w))$ for $w \in N^i(z)$, $i = 3, \ldots, e(z)$.

Now we can compute F by collecting all *track*–edges of the vertices of the set $V \setminus D(z, k)$. Obviously the above procedure runs in linear time.

Step (10). We use the BFS–tree rooted at z which was already computed in step (9). Let $b : V \to \mathbb{N}$ be the numbering of the vertices of G produced by BFS where $b(z) = 1$. Let S_1 (S_2) be the vertices of $N(z)$ ($N^2(z)$) which are endpoints of edges of F.

In what follows we explain a procedure looking for a pair of independent edges :

Consider the vertex x of S_1 with maximal b–number. By stepping through the neighbourhood of x we mark all vertices of S_1 which are either neighbours of x or fathers of neighbours of x in S_2 (cf. Figure 6 left).

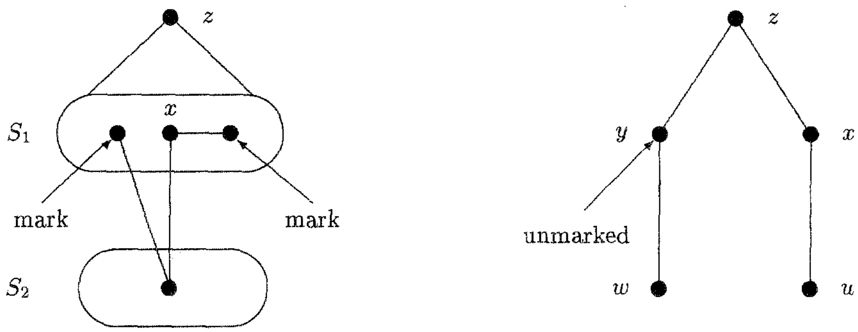


Fig. 6. Algorithm DHGDiam — Test for independent edges in F.

If there is an unmarked vertex $y \in S_1$ then there must be a neighbour w of y in S_2. We claim that the edges yw and xu, for some neighbour u of x in S_2, are independent (cf. Figure 6 right).
Indeed, since y is unmarked we must have $xw \notin E$ and $xy \notin E$. Since $b(x) > b(y)$, $x = f(u)$ and $y = f(w)$ the rules of BFS imply $uy \notin E$ (if $uy \in E$ then $f(u) = y$). Now $uw \notin E$ for otherwise the set $\{z, x, y, w, u\}$ induces a cycle of length five. Therefore, edges yw and xu are independent.

Now assume that all vertices of S_1 are marked. Then x cannot be an endpoint of a pair of independent edges. So we delete x from S_1 and all neighbours of x of S_2. We repeat the above procedure until we get a pair of independent edges or S_1 is empty.

Since processing a vertex x of S_1 takes $O(deg(x))$ the total running time of step (10) is linear.

Summarizing the above we get

Theorem 25. *For distance–hereditary graphs the diameter and a diametral pair of vertices can be computed in linear time.*

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