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On link diameter of a simple rectilinear polygon

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Abstract

The rectilinear link distance between two points inside a simple rectilinear polygon P is defined to be the minimum number of edges of a path consisting of axis-parallel segments lying inside P. The link diameter of P is the maximum link distance between two points of P. We present an $O(n)$ time algorithm for computing the link diameter of a simple rectilinear polygon P, where n is the number of vertices of P. This improves the previous $O(nlogn)$ time algorithm.

1 Introduction

A path π between two points x and y in a simple polygon P is a polygonal line inside P connecting x and y. The length of π is the number of line segments in this path. The link distance between two points in a simple polygon P is the length of a minimum length path connecting them [11]. This distance is used to model problems concerned with robot motion planning or broadcasting problems, and there is substantial literature on such issue (see [11,12,3-8,2]). Recently, a number of classical shortest path problems for the link measure of distance have been solved [4-8, 11,12]. In particular, a linear time algorithm for computing the link distance between any pair of points [11,12], an $O(nlogn)$ time algorithm for calculating the link diameter [11] and an $O(nlogn)$ algorithm for finding the link center of a simple n-vertex polygon [6,7] have been developed.

De Berg [3] consider some of these problems for rectilinear link distance inside a simple rectilinear polygon P (i.e., a simple polygon having all edges axis-parallel). A rectilinear path π is a polygonal line consisting of axis-parallel segments lying inside P. The length of the path, as in the case of simple polygons, is the number of segments in the path. The rectilinear link distance $d(x, y)$ between two points x and y of a simple rectilinear polygon P is the length of a minimum length rectilinear path connecting them [3]. The *link diameter* $d(P)$ of P is the maximum link distance between two points of P. Besides the algorithm for calculating the rectilinear link distance between any two given points within P, De Berg [3] presents an algorithm for computing the link diameter of P in time $O(nlogn)$, using a divide-and-conquer approach.

In this paper we study the properties of link diameters of the rectilinear polygons. We prove that $d(P) \geq 2r(P)-2$, where $r(P)$ is a radius of P (for link diameter of simple polygons a similar inequality was established in [8]). Using this fact we improve the De Berg algorithm and present an optimal algorithm for computing the link diameter of a simple rectilinear polygon without applying a divide-and-conquer method. Our algorithm essentially uses the linear algorithm for computing the link central point presented in [2] and the merging step of De Berg algorithm which runs in linear time too. Throughout in our paper, let P be a simple rectilinear polygon with n sides.

2 Properties of a rectilinear link distance

In this section we recall a few results about properties of rectilinear link distance, used in our algorithm. An axis-parallel segment is called a *cut* of P if it connects two sides of P and lies entirely inside P. By a *maximal cut* we will mean any maximal by inclusion cut of a polygon P. For a cut segment c and a point x of P the (rectilinear link) distance from x to c be defined as the distance from x to a closest point on c:

$$d(x, c) = \min\{d(x, p) : p \in c\} = d_x.$$

By a *farthest neighbor* of a cut c we will mean a point of P whose link distance from c is maximal. The distance from c to any its farthest neighbors is called the *eccentricity* $e(c)$ of c.

Following [3], let $c(x,d)$ be the part of c that can be reached from x with a (rectilinear) path π of length d such that the last segment of π is perpendicular to c. If we allow the first link of the path to have length zero, we always have $c(x,d_x) \subset c(x,d_x+1)$. Notice also that for any integer d the set $c(x,d)$ is a segment. The cut c *separates* two points x and y if x and y lie in different subpolygons defined by c.

Lemma 1 ([3]) *Let the cut c separates two points x and y, and let $d(x,c)=d_x$ and $d(y,c)=d_y$. Then we have*

$$d(x,y) = d_x + d_y + \Delta$$

where

$$\Delta = \begin{cases} -1 & \textit{if} \quad c(x,d_x) \bigcap c(y,d_y) \neq \emptyset \\ \ \ 0 & \textit{if} \quad c(x,d_x) \bigcap c(y,d_y) = \emptyset \ \textit{and} \\ & \qquad c(x,d_x+1) \bigcap c(y,d_y) \neq \emptyset \ \textit{or} \\ & \qquad c(x,d_x) \bigcap c(y,d_y+1) \neq \emptyset \\ +1 & \textit{otherwise} \end{cases}$$

Let s be any vertical or horizontal segment of a polygon P. The *visibility region* $VR(s)$ of s in P consists of the set of all points z in P visible from s, i.e. there is a cut of P perpendicular to s which passes through z.

Lemma 2 *Let c', c'' be the cuts perpendicular to the cut c and let the point z lies in the region R bounded by these cuts. If $d(z,c') < d(z,c) = k$ then $d(z,c'') = k+1$.*

Proof. Assume for example that c is a horizontal cut. Let s be the part of c which bound the region R. Then the point z lies in some pocket of R (by a *pocket* we will mean a connected component of R defined by a side of the visibility region $VR(s)$ of a segment s in R). Let c^* be the cut of P that separates this pocket from the rest of the region R. Evidently, $d(z,c) = d(z,c^*)+1$. Since z and c' lie in different subpolygons defined by c^* and $d(z,c^*) = d(z,c')$ then by Lemma 1 we deduce that for any point $x \in c'(z,k-1)$ we have $c^*(x,1) \bigcap c^*(z,k-1) \neq \emptyset$. This is possible only if $c^*(t,2) = \emptyset$ for any point $t \in c''$. By Lemma 1 we immediately obtain that $d(z,c'') = k+1$.

Lemma 3 ([2]) *For any cut c of P among its farthest neighbors there exists at least one vertex of a polygon P.*

Given a point $x \in P$, its *farthest neighbor* is a point of P whose link distance from x is the maximum over all other points of the polygon P. The distance from x to any its farthest neighbor is called the *eccentricity* $e(x)$ of x. The *radius* $r(P)$ is the minimum eccentricity of a point in P. The *center* $C(P)$ of P is the collection of all *central points*, i.e. points whose eccentricities are equal to $r(P)$.

Lemma 4 ([2]) *For any point x of P among its farthest neighbors there exists at least one vertex of a polygon P.*

Now we establish the relation between the link radius and link diameter of a rectilinear polygon P.

Theorem 1 *For any simple rectilinear polygon P*

$$2r(P) - 2 \leq d(P) \leq 2r(P).$$

Proof. Let c be the maximal cut with minimal eccentricity. Note that $r(P) - 1 \leq e(c) \leq r(P)$. If $e(c) = r(P)$ then the intersection $\bigcap \{c(v, r(P)) : v \text{is a vertex with} d(v, c) \geq r(P) - 1\}$ is empty, otherwise the cut perpendicular to c which passes through the point from this intersection has eccentricity $r(P) - 1$. By Helly property there exist two disjoint segments $c(v, r(P))$ and $c(w, r(P))$. Let y be the point of c that separates these segments. Then the cut c' perpendicular to c and which passes through y separates v and w and $d(v, c') \geq r(P)$, $d(w, c') \geq r(P)$. By Lemma 1 we obtain the inequality $d(v, w) \geq 2r(P) - 1$.

Now assume that the minimal eccentricity of cuts is $r(P) - 1$. For any point $x \in P$ let $V(x)$ be the set of all vertices v of P for which $d(v, x) \leq r(P) - 1$. Let x^* be the point with maximal set $V(x^*)$ among the points which lie on cuts with minimal eccentricity. By invoking the definition of radius and from Lemma 4 there exists a vertex w with $d(x^*, w) = r(P)$. Let c be the maximal cut of P with $e(c) = r(P) - 1$

which passes through the point x^*. Note that the last link of some shortest path from w to x^* belong to c, and x^* is a point of c which belongs to maximal number of metric projections of vertices situated at distance $r(P)-1$ from c. We claim that there exists a vertex $z \in V(x^*)$, such that

$$c(z, r(P)-1) \bigcap c(w, r(P)-1) = \emptyset.$$

To show this first observe that $x^* \notin c(w, r(P)-1)$ and $x^* \in \bigcap \{c(v, r(P)-1) : v \in V(x^*)\}$. Hence, if $c(w, r(P)-1)$ intersects any segment $c(v, r(P)-1)$, $v \in V(x^*)$, then by Helly Theorem there exists a point

$$p \in \bigcap \{c(v, r(P)-1) : v \in V(x^*)\} \bigcap c(w, r(P)-1).$$

Since $V(p) \supset V(x^*) \cup \{w\}$ we obtain a contradiction with our choice.

Let c' be a maximal cut perpendicular to c and which separates the segments $c(z, r(P)-1)$ and $c(w, r(P)-1)$. Denote by x the common point of cuts c and c'. Let P' and P'' be the subpolygons defined by c' and suppose that $z \in P'$, $w \in P''$. Since $x \notin c(z, r(P)-1) \bigcup c(w, r(P)-1)$ then $d(z, c') \geq r(P)-1$ and $d(w, c') \geq r(P)-1$. If

$$e(c') \geq r(P) \quad \text{or} \quad c'(w, r(P)-1) \bigcap c'(z, r(P)-1) = \emptyset$$

then by Lemma 1 we obtain $d(P) \geq 2r(P)-2$. So assume that $e(c') = d(w, c') = r(P)-1$ and $c'(w, r(P)-1) \bigcap c'(z, r(P)-1) \neq \emptyset$.

For any vertex v put

$$c'_+(v) = \{p \in c' : d(v, p) \leq r(P)-1\}.$$

Note that if the set $c'_+(v)$ is non-empty then either $c'_+(v) = c'(v, r(P)-1)$ or $c'_+(v) = c'$. We claim that either $d(P) \geq 2r(P)-2$ or for any vertex $v \in V(x^*)$ we have $c'_+(v) \cap c'_+(w) \neq \emptyset$. If $x \in c(v, r(P)-1)$ then $d(v, c') \leq r(P)-2$ and therefore $c'_+(v) = c'$. So assume that $x \notin c(v, r(P)-1)$, i.e. $d(v, c') \geq r(P)-1$. If $v \in P'$ then $d(v, c') = r(P)-1$ and $c'(v, r(P)-1) \bigcap c'(w, r(P)-1) \neq \emptyset$, otherwise by Lemma 1 we have $d(v, w) \geq 2r(P)-2$. Now suppose that $v \in P''$. Since $x \notin c(v, r(P)-1)$ and $d(v, x^*) \leq r(P)-1$

this yields that $d(v,c) = r(P)-2$. Hence $d(v,c') = r(P)-1$ and $x \in c'(v, r(P)-1)$. Let $c'_+(v) \cap c'_+(w) = \emptyset$ and c^* be a cut perpendicular to c' that separates the segments $c'_+(v) = c'(v, r(P)-1)$ and $c'_+(w) = c'(w, r(P)-1)$. Denote by y the intersection of cuts c' and c^*. Since $y \notin c'(v, r(P)-1) \bigcup c'(w, r(P)-1)$ then $d(v,c^*) \geq r(P)-1$ and $d(w,c^*) \geq r(P)-1$. This is possible only if the cut c^* separates the cut c and the vertex w, otherwise $d(v,c^*) < r(P)-1$. So the vertex v lies in the region bounded by the cuts c, c' and c^* and $d(v,c) = r(P)-2, \quad d(v,c') = r(P)-1$. By Lemma 2 we conclude that $d(v,c^*) \geq r(P)$. Using Lemma 1 for cut c^* and vertices v and w we obtain

$$d(v,w) \geq d(v,c^*) + d(w,c^*) - 1 \geq 2r(P) - 2.$$

This proves the claim.

Finally we will show that either $d(P) \geq 2r(P)-2$ or for any vertices $v', v'' \in V(x^*)$ we have $c'_+(v') \cap c'_+(v'') \neq \emptyset$. It is enough to consider only the case when $d(v',c') = d(v'',c') = r(P)-1$ and vertices v' and v'' lie in the same subpolygon P' defined by the cut c', otherwise we immediately obtain the required property. Suppose that $c'_+(v') \cap c'_+(v'') = \emptyset$ and let c'' be the maximal cut perpendicular to c' which separates the segments $c'_+(v') = c'(v', r(P)-1)$ and $c'_+(v'') = c'(v'', r(P)-1)$. If $c'' = c$ then we obtain that the cut c' separates the vertices v', v'' and the vertex w. Since v' and v'' lie in different subpolygons defined by c then this cut separates the vertex w and one of the vertices v' and v'' too. Assume for example that c separates vertices w and v'. Since $c(v', r(P)-1) \bigcap c(w, r(P)-1) = \emptyset$ then by Lemma 1 $d(v', r(P)-1) \geq 2r(P)-2$. So $c \neq c''$. Assume for example that the cut c and the vertex v'' lie in different subpolygons defined by c''. Now recall that $d(v', x^*) \leq r(P)-1$ and $d(v'', x^*) \leq r(P)-1$. According to Lemma 1 this is possible only if $d(v'', c'') = r(P)-1$ and $c''(x^*, 1) \cap c''(v'', r(P)-1) \neq \emptyset$. Let y^* be the point from the last intersection. Observe that the whole rectangle bounded by the cuts c, c', c'' and the segment x^*y^* is contained in P. If $d(v', c) = r(P)-1$ then $d(v', x^*) = r(P)-1$ and the last link of any shortest path from v' to x^* is a part of a segment x^*y^*. In this case we obtain a path of length $r(P)-2$ from v' to some point z^* of a cut c', contradicting our assumption that $d(v',c')=r(P)-1$. On the other hand if $d(v',c)=r(P)-2$ then by Lemma 2 we deduce that $d(v',c'')\geq r(P)$. Since $d(v'',c'')\geq r(P)-1$ then by Lemma 1 we obtain $d(v',v'')\geq 2r(P)-2$.

Thus if we assume that $d(P)<2r(P)-2$ then for any vertices $v',v''\in V(x^*)$ we have

$$c'_+(v')\cap c'_+(w)\neq\emptyset,\quad c'_+(v')\cap c'_+(v'')\neq\emptyset.$$

By Helly Theorem there exists a point $x^{**}\in\bigcap\{c'_+(v):v\in V(x^*)\}\cap c'_+(w)$. Hence $V(x^{**})\supset V(x^*)\cup\{w\}$, contradicting to choice of a point x^*. This finishes the proof []

Remark. The similar relation between link radius and link diameter of a simple polygon was established by Lenhart et all [8]. Unfortunately, their elegant proof using the Molnar Theorem for cells can not be transfered to rectilinear polygons, since in the rectilinear link metric the intersection of two disks may be not connected.

3 The algorithm

De Berg [3] gave a linear time algorithm which for any cut c and subpolygons P' and P'' defined by c compute the value $\max\{d(v,w):v\in P',w\in P''\}$. On the other hand, in [2] the authors present a linear algorithm for finding the central point and link radius of a rectilinear polygon P. These algorithms, the results of the previous section and of paper [3] lead to the following algorithm for computing the link diameter of P.

For cut c of a polygon P put

$$\begin{aligned} D(c) &= \{v \text{ vertex of } P: d(v,c)=e(c)\},\\ D^+(c) &= \{v \text{ vertex of } P: d(v,c)\geq r(P)-1\}. \end{aligned}$$

By a *diametral pair* of vertices we will mean any two vertices v,w with $d(v,w)=d(P)$.

Algorithm. *Computing the link diameter of a simple rectilinear polygon P.*

1. find a link central point x and radius $r(P)$ of polygon P;
2. find the eccentricity of a maximal cut c, which passes through the point x;
3. **if** $(e(c) = r(P) - 1)$ **then begin**
4.   **if** $(\bigcap \{c(z, r(P)) : z \in D^+(c)\} \neq \emptyset)$ **then begin**
5.   find a maximal cut $c^\perp$ perpendicular to c which passes through the point from $\bigcap \{c(z, r(P)) : z \in D^+(c)\}$;
6.   let P' and P'' be the subpolygons defined by c and $P'_\perp$ and $P''_\perp$ be the subpolygons defined by $c^\perp$;
7.   compute $M = \max\{d(v, w) : v \in P', w \in P''\}$;
8.   compute $M_\perp = \max\{d(v, w) : v \in P'_\perp, w \in P''_\perp\}$;
9. $d(P) = \max\{2r(P) - 2, M, M_\perp\}$; **stop**

            end;

10.   find two disjoint segments $c(v, r(P))$ and $c(w, r(P))$;
11. $d(P) = d(v, w) = 2r(P) - 1$; **stop**

        end;

            $(*e(c) = r(P)*)$

12. **if** $(\bigcap \{c(z, r(P)) : z \in D^+(c)\} \neq \emptyset)$ **then begin**
13.   find a maximal cut $c^\perp$ perpendicular to c which passes through the point from $\bigcap \{c(z, r(P)) : z \in D^+(c)\}$;
14. $c = c^\perp$; $(*e(c) = r(P) - 1*)$
15.   repeat steps (3)–(11)

            end;

16. find two disjoint segments $c(v, r(P))$ and $c(w, r(P))$;
17. $M_0 = d(v, w)$;
18. find a maximal cut $c^\perp$ perpendicular to c which passes through the point x;
19. let P' and P'' be the subpolygons defined by c and $P'_\perp$ and $P''_\perp$ be the subpolygons defined by $c^\perp$;
20. compute $M = \max\{d(v, w) : v \in P', w \in P''\}$;
21. compute $M_\perp = max\{d(v, w) : v \in P'_\perp, w \in P''_\perp\}$;
22. $d(P) = max\{M_0, M, M_\perp\}$;
23. **if** $(d(P) = 2r(P) - 1)$ **then begin**

24. find a pair v, w of vertices from $P_i \quad (i = 1, 2, 3, 4)$ such that
$d(v, c) + d(w, c) = d(v, c^{\perp}) + d(w, c^{\perp}) = 2r(P) - 1$ and
$c(v, r(P)) \bigcap c(w, r(P)) = \emptyset$;

25. **if** (such pair of vertices v, w exists for some index i) **then**
$d(P) = 2r(P) = d(v, w)$
end;

Theorem 2 *The rectilinear link diameter of a simple rectilinear polygon P on n vertices can be computed in $O(n)$ time.*

Proof. We first prove the correctness of our algorithm. So we distinguish two cases.

Case 1. $e(c) = r(P) - 1$.

From Theorem 1 it follows that either $d(P) = 2r(P) - 2$ or $d(P) = 2r(P) - 1$. So, if we find two disjoint segments $c(v, r(P))$ and $c(w, r(P))$ (see step (10)) then we immediately obtain that $d(v, w) \geq 2r(P) - 1$ and therefore $d(P) = d(v, w) = 2r(P) - 1$. To show this consider any cut c^{+} perpendicular to c which separates these segments. Since $d(v, c^{+}) \geq r(P)$, $\quad d(w, c^{+}) \geq r(P)$ and c^{+} separates vertices v and w then by Lemma 1 we obtain the required inequality.

Now suppose that $\bigcap \{c(v, r(P)) : v \in D^{+}(c)\} \neq \emptyset$ (steps (4)–(9)). For a proof of correctness of these steps it is enough to show that if $d(P) = 2r(P) - 1$ then $\max\{M, M_{\perp}\} = 2r(P) - 1$. Suppose this fails and pick any pair v, w of diametral vertices. Without loss of generality assume that $v, w \in P' \cap P'_{\perp}$. Since $e(c) = e(c^{\perp}) = r(P) - 1$ then

$$d(v, c) = d(v, c^{\perp}) = d(w, c) = d(w, c^{\perp}) = r(P) - 1$$

and

$$\begin{aligned} c(v, r(P) - 1) \bigcap c(w, r(P) - 1) &= \emptyset, \\ c^{\perp}(v, r(P) - 1) \bigcap c^{\perp}(w, r(P) - 1) &= \emptyset. \end{aligned}$$

Let c' be the cut perpendicular to $c^{\perp}$ which separates the segments $c^{\perp}(v, r(P) - 1)$ and $c^{\perp}(w, r(P) - 1)$ and suppose that w and c lie in different subpolygons defined by c'. Since $d(w, c) = r(P) - 1$ then by

Lemma 1 $d(w,c') = r(P)-1$. Moreover, for any point $x \in c(w,r(P)-1)$ we have $d(x,c') = 1$ and $c'(x,1) \bigcap c'(w,r(P)-1) \neq \emptyset$. Pick any point x' from this intersection. Denote by c'' the cut which passes through the points x and x'. Then $d(w,c'') = r(P)-2$. Observe also that the rectangle bounded by the cuts $c, c', c^{\perp}$ and c'' is contained in the polygon P. Hence the cut c'' separates the vertex v and the cut $c^{\perp}$. Since $d(v,c^{\perp}) = r(P)-1$ then $d(v,c'') \leq r(P)-1$. This yields

$$d(v,w) \leq r(P) - 1 + r(P) - 2 + 1 = 2r(P) - 2,$$

a contradiction.

Case 2. $e(c) = r(P)$.

It is enough to consider only the case when on step (12) we obtain an empty intersection. Then as in preceding case we can show that for vertices $v, w \in D^{+}(c)$ with disjoint segments $c(v,r(P))$ and $c(w,r(P))$ the inequality $d(v,w) \geq 2r(P)-1$ holds. So $d(P) \geq 2r(P)-1$.

Further, according to our algorithm we define the maximal cut $c^{\perp}$ perpendicular to c which passes through the central point of P. Let P_1, P_2, P_3 and P_4 be the subpolygons defined by cuts c and $c^{\perp}$. Put

$$\begin{aligned} M = \ & \max\{d(v,w) : \ v \text{ and } w \text{ lie in different subpolygons defined by } c\}. \\ M_{\perp} = \ & \max\{d(v,w) : \ v \text{ and } w \text{ lie in different subpolygons defined by } c^{\perp}\}. \end{aligned}$$

Now assume that $\max\{M, M_{\perp}\} = 2r(P)-1$, however $d(P) = 2r(P)$. Next we claim that for vertices $v, w \in P_i$ we have $d(v,w) = 2r(P)$ if and only if

$$d(v,c) + d(w,c) = d(v,c^{\perp}) + d(w,c^{\perp}) = 2r(P) - 1$$

and either $c(v,r(P)) \cap c(w,r(P)) = \emptyset$ or $c^{\perp}(v,r(P)) \cap c^{\perp}(w,r(P)) = \emptyset$. First assume that $v, w \in P_1$ and $d(v,w) = 2r(P)$. Observe that

$$c(v,r(P)) \cap c(w,r(P)) = \emptyset, \qquad c^{\perp}(v,r(P)) \cap c^{\perp}(w,r(P)) = \emptyset,$$

otherwise if say $z \in c(v, r(P)) \cap c(w, r(P))$ then from two paths of length $r(P)$ from z to vertices v and w we obtain a path of length $2r(P) - 1$ between v and w. On the other hand, since the point $x = c \cap c^{\perp}$ is a central point and c and $c^{\perp}$ are maximal cuts then

$$\min\{d(v,c), d(v,c^{\perp}\} = \min\{d(w,c), d(w,c^{\perp})\} = r(P) - 1.$$

Suppose that $d(v,c) = r(P) - 1$. Since $d(v,w) = 2r(P)$ then $d(w,c) = r(P)$ and therefore $d(w,c^{\perp}) = r(P) - 1, \quad d(v,c^{\perp}) = r(P)$.

For a proof of converse implication assume that $v, w \in P_1$ be such vertices that $c(v, r(P)) \cap c(w, r(P)) = \emptyset$ and

$$d(v,c) + d(w,c) = d(v,c^{\perp}) + d(w,c^{\perp}) = 2r(P) - 1.$$

Let for example $d(v,c) = r(P) - 1, \quad d(w,c) = r(P)$. Since $d(w,x) \leq r(P)$ and the first link of any path from x to w is a segment of one of the cuts c or $c^{\perp}$ then $d(w,c^{\perp}) = r(P) - 1$ and so $d(v,c^{\perp}) = r(P)$. Let c' be the cut perpendicular to c which separates the disjoint segments $c(v, r(P))$ and $c(w, r(P))$. Then c' separates the vertices v and w too and $d(v,c') \geq r(P), \quad d(w,c') \geq r(P)$. Since $d(w,c^{\perp}) = r(P) - 1$ then by Lemma 1 this is possible only if the vertex w lies in the region bounded by the cuts $c^{\perp}$, c and c'. According to Lemma 2 we have $d(w,c') \geq r(P) + 1$ and by the same Lemma 1

$$d(v,w) \geq r(P) + r(P) + 1 + \Delta \geq 2r(P).$$

Now we pay attention to the complexity of the Algorithm. As we earlier mention steps (1), (7), (8) and (20), (21) can be performed in $O(n)$ time. The finding of a cut or a maximal cut can be done in $O(n)$ time too. The eccentricity of any cut can be computed in linear time ; see [3, p.22]. Moreover, from the results of papers [1] and [3] follows that for all vertices from the set $D^{+}(c)$ the segments $c(z, r(P))$ can be computed in linear time in total. On steps (4), (10), (12) and (16) we either establish that the intersection of a family of segments is nonempty or find a pair of disjoint segments. This operation take $O(n)$ time; see, for example, [9,10]. The pair of vertices $v, w \in P_i$ on step

(24) may be found in the following way. Assume for example that c be a horizontal cut. Among the vertices $u \in P_i$ for which $d(u,c) = r(P)$ choose a vertex v such that the right end of a segment $c(v, r(P))$ is minimal. Now among the vertices $u \in P_i$ with $d(u,c) = d(u,c^{\perp}) - 1 = r(P) - 1$ choose a vertex w such that left end of the segment $c(w, r(P))$ is maximal. As we already proof, if $c(v, r(P)) \cap c(w, r(P)) = \emptyset$ for such pair v, w of a some subpolygon P_i then $d(P) = 2r(P)$. Otherwise, we conclude that $d(P) = 2r(P) - 1$. From this it follows that the total running time of the Algorithm is $O(n)$. []

Note added to proof. After the submission of this paper we learned a paper of B.J. Nilsson and S. Schuierer "Computing the rectilinear link diameter of a polygon". In this paper the linear time algorithm for finding the link diameter was also given. Their approach completely differs from our and is based on the divide-and-conquer paradigm.

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