# 1 Rasterization

## 1.1 Pixels

**Pixel:** Intensity or colour sample.

**Raster Image:** Rectangular grid of pixels.

**Rasterization:** Conversion of a primitive's geometric representation into

- A set of pixels.
- An intensity or colour for each pixel (**shading**, **antialiasing**).

For now, we will assume that:

- The **background color** is white.
- The **foreground color** is black.
- All pixels are initially **cleared** to white.
- Algorithms **set** pixels to the foreground colour: colour them black.

## 1.2 Pixel Grids

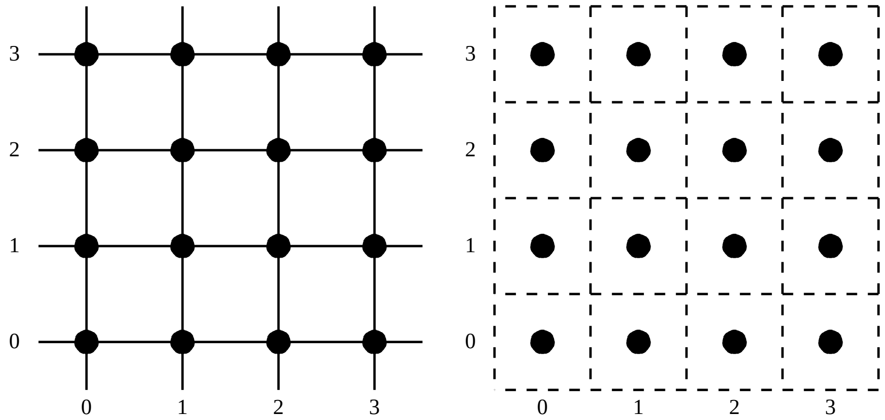
**Pixel Centres:** Address pixels by integer coordinates $(i, j)$

**Pixel Centre Grid:** Set of lines passing through pixel centers.

**Pixel Domains:** Rectangular semi-open areas surrounding each pixel center:

$$\mathcal{D}_{i,j} = [i - 1/2, i + 1/2) \times [j - 1/2, j + 1/2).$$

**Pixel Domain Grid:** Set of lines formed by domain boundaries.




Pixel Centre Grid    Pixel Domain Grid

## 1.3 Specifications and Representations

Each rendering primitive (point, line segment, polygon, etc) needs both

- A **geometric** specification, usually "**calligraphic**"
- A **rasterized** pixel representation

Standard device-level geometric specifications include:

**Point:** $A = (x_A, y_A) \in \mathbb{R}^2$.

**Line Segment:** $\mathcal{L}(AB)$ specified with points $A$ and $B$.
$\mathcal{L}(AB)$ is set of all collinear points between $A$ and $B$.

**Polygon:** $\mathcal{P}(A_1A_2\ldots A_n)$ specified with ordered list of points $A_1A_2\ldots A_n$.
$\mathcal{P}(A_1A_2\ldots A_n)$ is set of points inside a piecewise linear boundary;
edges are given by $A_i$ to $A_{i+1}$ and $A_n$ to $A_1$.
**This "list of points" specification is flawed and imprecise** ...
**A better definition will be given later.**

## 1.4 Line Segments

Let $\mathcal{L}(AB) = \{P \in \mathbb{R}^2 | P = (1-t)A + tB, t \in [0,1]\}$.

**Problem:** Given a line segment $\mathcal{L}(AB)$ specified by points $A$ and $B$,

**Decide:** Which pixels should be **set** to represent $\mathcal{L}(AB)$.

**Desired Properties:** Rasterization of line segment **should**:

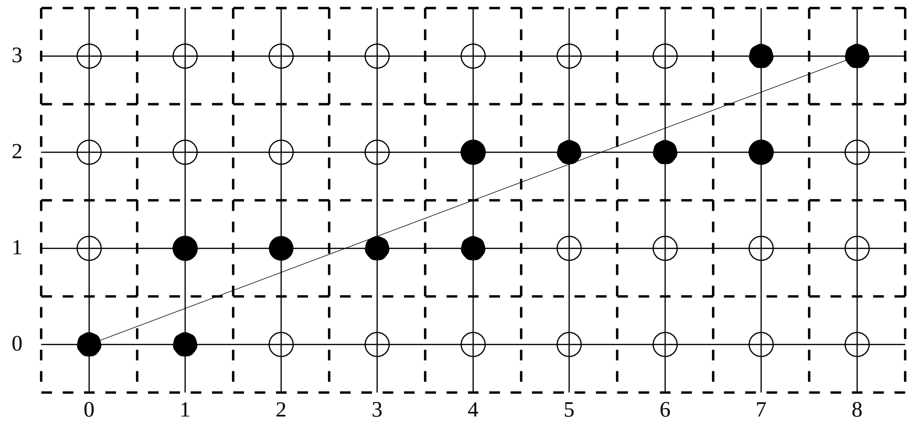
1. Appear as straight as possible
2. Include pixels containing $A$ and $B$
3. Have relatively constant intensity
   (ie, all parts should be the same brightness)
4. Have an intensity per unit length that is independent of slope
5. Be symmetric
6. Be generated efficiently

♠ *Readings: Hearn and Baker, Section 3-2.*

### 1.5 Line Segment Pixel Representations

1. Given $AB$, choose a set of pixels $L_1(AB)$ given by

$$L_1(AB) = \{(i,j) \in \mathbb{Z}^2 | \mathcal{L}(AB) \cap \mathcal{D}_{i,j} \neq \emptyset\}.$$

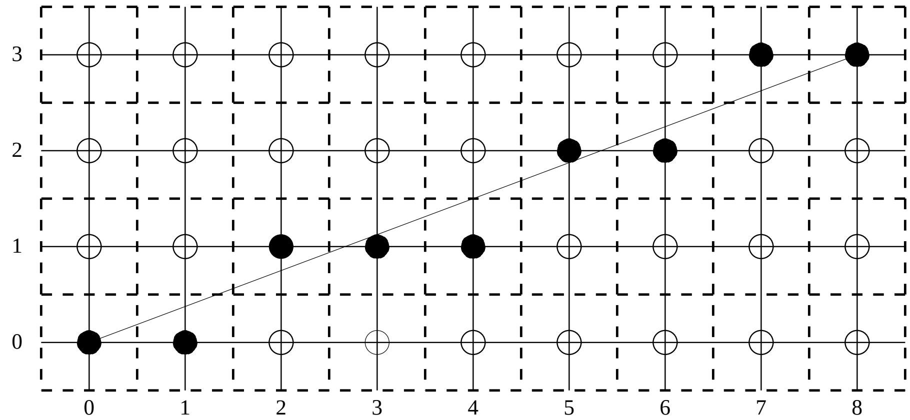



Unfortunately, this results in a very blotchy, uneven looking line.

2. Given $AB$, choose a set of pixels $L_2(AB)$ given by

$$L_2(AB) = \begin{cases} |x_B - x_A| \geq |y_B - y_A| \longrightarrow \\ \quad \{(i,j) \in \mathbb{Z}^2 | (i,j) = (i,[y]), (i,y) \in \mathcal{L}(AB)\} \\ \quad \cup ([x_A],[y_A]) \cup ([x_B],[y_B]). \\ |x_B - x_A| < |y_B - y_A| \longrightarrow \\ \quad \{(i,j) \in \mathbb{Z}^2 | (i,j) = ([x],j), (x,j) \in \mathcal{L}(AB)\} \\ \quad \cup ([x_A],[y_A]) \cup ([x_B],[y_B]). \end{cases}$$

Where $[z] = \lfloor z + 1/2 \rfloor$,
and $\lfloor w \rfloor$ is the greatest integer less than or equal to $w$.




Not symmetric … oh, well.

### 1.6 Line Equation Algorithm

Assume $\Delta y \leq \Delta x$, $m = \Delta y / \Delta x \leq 1$.

Based on the line equation $y = mx + b$, we can derive:

```
LineEquation (int xA, yA, xB, yB)
    float m, b;
    int xi, dx;

    m = (yB - yA)/(xB - xA);
    b = yA - m*xA;
    if ( xB - xA > 0 ) then dx = 1;
                       else dx = -1;
    for xi = xA to xB step dx do
        y = m*xi + b;
        SetPixel( xi, [y] );
    endfor
```

Problems with line equation algorithm:

- One pixel per column so lines of slope $> 1$ have gaps
- Vertical lines cause divide by zero

To fix these problems when $m > 1$,

- Use $x = m^{-1}(y - b)$
- Loop over $y$ instead of $x$

### 1.7 Discrete Differential Analyzer

**Observation:** Roles of $x$ and $y$ are symmetric ...

- Change roles of $x$ and $y$ if $|y_B - y_A| > |x_B - x_A|$

**Observation:** Multiplication by $m$ inside loop ...

- The value of $m$ is constant for all iterations.
- Can reduce computations inside loop:
  compute $y_{i+1}$ incrementally from $y_i$:

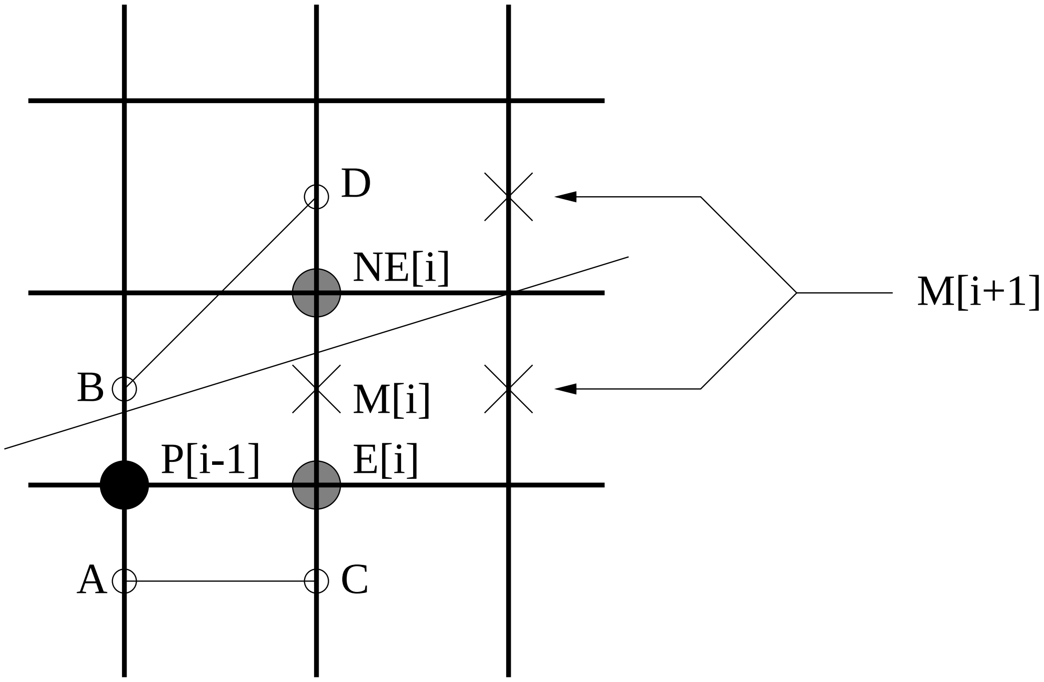
$$y_{i+1} = m(x_i + 1) + b = y_i + m$$

```
DDA (int xA, yA, xB, yB)
    int length, dx, dy, i;
    float x,y,xinc,yinc;
```

```
dx = xB - xA;
dy = yB - yA;

length = max ( |dx| > |dy| );

xinc = dx/length;  # either xinc or yinc is -1 or 1
yinc = dy/length;

x = xA; y = yA;
for i=0 to length do
    SetPixel( [x], [y] );
    x += xinc;
    y += yinc;
endfor
```

### 1.8 Bresenham's Algorithm

- Completely integer;
- Will assume (at first) that $x_A, y_A, x_B, y_B$ are also integer.
- Only addition, subtraction, and shift in inner loop.
- Originally for a pen plotter.
- "Optimal" in that it picks pixels closest to line, i.e. $L_2(AB)$.
- Assumes $0 \leq (y_B - y_A) \leq (x_B - x_A) \leq 1$ (i.e., slopes between 0 and 1).
- Use reflections and endpoint reversal to get other slopes: 8 cases.




- Suppose we know at step $i-1$ that pixel $(x_i, y_i) = P_{i-1}$ was chosen.
  Thus, the line passed between points $A$ and $B$.
- Slope between 0 and 1 $\Rightarrow$
  line must pass between points $C$ and $D$ at next step $\Rightarrow$
  $E_i = (x_i + 1, y_i)$ and $NE_i = (x_i + 1, y_i + 1)$ are only choices for next pixel.
- If $M_i$ above line, choose $E_i$;
- If $M_i$ below line, choose $NE_i$.
- Implicit representations for line:

$$y = \frac{\Delta y}{\Delta x} x + b$$

$$F(x, y) = \underbrace{(2\Delta y)}_{Q} x + \underbrace{(-2\Delta x)}_{R} y + \underbrace{2\Delta x b}_{S} = 0$$

where

$$\begin{aligned} \Delta x &= x_B - x_A \\ \Delta y &= y_B - y_A \\ b &= y_A - \frac{\Delta y}{\Delta x} x_A \Rightarrow S = 2\Delta x y_A - 2\Delta y x_A \end{aligned}$$

Note that

1. $F(x, y) < 0 \Rightarrow (x, y)$ **above** line.
2. $F(x, y) > 0 \Rightarrow (x, y)$ **below** line.
3. $Q, R, S$ are all integer.

- The mystery factor of 2 will be explained later.
- Look at $F(M_i)$. Remember, $F$ is 0 if the point is on the line:
  - $F(M_i) < 0 \Rightarrow M_i$ **above** line $\Rightarrow$ choose $P_i = E_i$.
  - $F(M_i) > 0 \Rightarrow M_i$ **below** line $\Rightarrow$ choose $P_i = NE_i$.
  - $F(M_i) = 0 \Rightarrow$ arbitrary choice; consider choice of pixel domains ...
- We'll use $d_i = F(M_i)$ as a **decision variable.**
- Can compute $d_i$ incrementally with integer arithmetic.
- At each step of algorithm, we **know** $P_{i-1}$ and $d_i$ ...
- Want to **choose** $P_i$ and **compute** $d_{i+1}$
- Note that

$$\begin{aligned} d_i &= F(M_i) = F(x_{i-1} + 1, y_{i-1} + 1/2) \\ &= Q \cdot (x_{i-1} + 1) + R \cdot (y_{i-1} + 1/2) + S \end{aligned}$$

- If $E_i$ is chosen then

$$\begin{aligned} d_{i+1} &= F(x_{i-1} + 2, y_{i-1} + 1/2) \\ &= Q \cdot (x_{i-1} + 2) + R \cdot (y_{i-1} + 1/2) + S \\ &= d_i + Q \end{aligned}$$

- If $NE_i$ is chosen then

$$\begin{aligned} d_{i+1} &= F(x_{i-1} + 2, y_{i-1} + 1/2 + 1) \\ &= Q \cdot (x_{i-1} + 2) + R \cdot (y_{i-1} + 1/2 + 1) + S \\ &= d_i + Q + R \end{aligned}$$

- Initially, we have

$$\begin{aligned} d_1 &= F(x_A + 1, y_A + 1/2) \\ &= Qx_A + Ry_A + S + Q + R/2 \\ &= F(x_A, y_A) + Q + R/2 \\ &= Q + R/2 \end{aligned}$$

- Note that $F(x_A, y_A) = 0$ since $(x_A, y_A) \in \ell(AB)$.

- Why the mysterious factor of 2?
  It makes everything integer.

```
Bresenham (int xA, yA, xB, yB)
    int d, dx, dy, xi, yi, incE, incNE
    dx = xB - xA
    dy = yB - yA
    incE = dy<<1               /* Q */
    incNE = incE - dx<<1;      /* Q + R */
    d = incE - dx              /* Q + R/2 */
    xi = xA; yi = yA
    SetPixel( xi, yi )
    while ( xi < xB )
        xi++
        if ( d < 0 ) then /* Mi above L; choose E */
            d += incE
        else              /* Mi below L; choose NE */
            d += incNE
            yi++
        endif
        SetPixel( xi, yi )
    endwhile
```

- Some asymmetries from line rep, choice of pixel domains.
- Did we meet our goals?
  1. Straight as possible: yes, but depends on metric.
  2. Correct termination.
  3. Even distribution of intensity: yes, more or less, but:
  4. Intensity varies as function of slope.
     - Can't do better without gray scale.
     - Worst case: diagonal compared to horizontal
  5. Fast! (if integer math fast ... )
- Interaction with clipping?
- Subpixel positioning of endpoints?
- Variations that look ahead more than one pixel at once ...
- Variations that compute from both end of the line at once ...
- Similar algorithms for circles, ellipses ...
  (8 fold symmetry for circles)

♠ *Readings: Red book: 3.2.2, White book: 3.2.2.*