Examination of the performance of robust numerical methods for singularly perturbed quasilinear problems with interior layers.

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1 Continuous problem class

In this paper we examine the numerical performance of parameter–robust numerical methods [1] for the following class of quasilinear singularly perturbed boundary value problems: Let $\Omega^- := (0,d)$, $\Omega^+ := (d,1)$ and find $u_\varepsilon \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$\varepsilon u_\varepsilon'' + b(x,u)u_\varepsilon' = f, \quad \text{for all} \quad x \in \Omega^- \cup \Omega^+, \qquad \text{(1a)}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B, \qquad \text{(1b)}$$

$$b(x,u) = \begin{cases} b_1(u) = -1 + cu, & x < d \\ b_2(u) = 1 + cu, & x > d \end{cases}, \quad f(x) = \begin{cases} -\delta_1, & x < d \\ \delta_2, & x > d \end{cases} \qquad \text{(1c)}$$

$$-1 < u_\varepsilon(0) < 0, \quad 0 < u_\varepsilon(1) < 1, \quad 0 < c \le 1, \qquad \text{(1d)}$$

where δ_1, δ_2 are non–negative constants. Note the strict inequalities in (1d), which are imposed in order to ensure that the solution exhibits a standard shock layer, as opposed to a S-type layer.

In order to guarantee existence and uniqueness of the solution of the continuous problem, we need to impose additional conditions on the magnitudes of $\|f\|$ and the boundary values $|u_\varepsilon(0)|$, $|u_\varepsilon(1)|$. Further restrictions are required in the theoretical analysis in [4] to prove uniform in ε convergence of the numerical method described below. These conditions are stated in (4) and (7). A linear version of (1) was studied in [2], where a parameter–uniform numerical method based on a suitably designed piecewise–uniform mesh was shown to be parameter-uniform of essentially first order for a linear convection-diffusion problem with discontinuous data. The methodology in [2] was extended in [4] to the quasilinear problem (1) under the conditions (4) and (7).

Let $\mathbf{C}_1$ be the class of problems defined by (1),(3); $\mathbf{C}_2$ be the class of problems defined by (1),(4) and $\mathbf{C}_3$ be the class of problems defined by (1),(4) and (7). The proof in [4] restricts the problem to the smallest of these three classes $\mathbf{C}_3$. Figure 1 displays some typical solutions for some sample problems in $\mathbf{C}_3$. In this paper, we examine (via numerical experiments) the parameter-

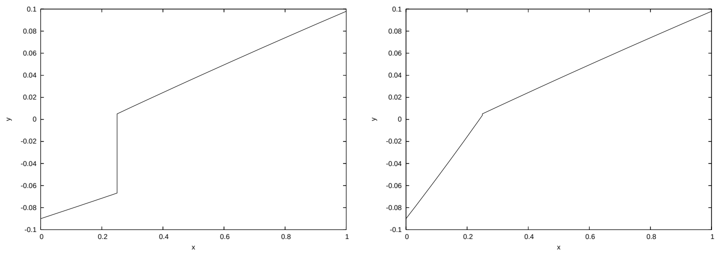

Fig. 1. Solution of (1) for sample problems in $\mathbf{C}_3$.

uniform performance of the numerical method under the weaker conditions $\mathbf{C}_1$ or $\mathbf{C}_2$.

The reduced solution $v_0 : [0,1] \rightarrow (-1,1)$ is defined to be the solution of the following nonlinear **first order** problem

$$b(v_0, x)v_0' = f,\ x \in \Omega^- \cup \Omega^+, \qquad v_0(0) = u_\varepsilon(0),\ v_0(1) = u_\varepsilon(1). \tag{2}$$

A unique reduced solution v_0 with the additional sign-pattern property of $v_0(x) < 0,\ x \in \Omega^-; \quad v_0(x) > 0,\ x \in \Omega^+$exists if the conditions [4]

$$\delta_1 d < -u_\varepsilon(0) + 0.5cu_\varepsilon^2(0),\ \delta_2(1-d) < u_\varepsilon(1) + 0.5cu_\varepsilon^2(1), \tag{3}$$

are satisfied by the data. For a unique solution of the full continuous problem to exist it suffices [4] that

$$\delta_1 d < -u_\varepsilon(0), \quad \delta_2(1-d) < u_\varepsilon(1) \tag{4a}$$

$$u_\varepsilon(1) - u_\varepsilon(0) < 1/c + \min\{\frac{\delta_1 d}{1 - cu_\varepsilon(0)}, \frac{\delta_2(1-d)}{1 + cu_\varepsilon(1)}\}. \tag{4b}$$

Note that (4a) implies (3) and hence $\mathbf{C}_3 \subset \mathbf{C}_2 \subset \mathbf{C}_1$.

2 Numerical method

The domain $\overline{\Omega}$ is subdivided into the four subintervals

$$[0, d - \sigma_1] \cup \quad [d - \sigma_1, d] \cup [d, d + \sigma_2] \cup [d + \sigma_2, 1], \tag{5a}$$

for some σ_1, σ_2 that satisfy $0 < \sigma_1 \leq \frac{d}{2}$, $0 < \sigma_2 \leq \frac{1-d}{2}$. On each of the four subintervals a uniform mesh with $\frac{N}{4}$ mesh-intervals is placed. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N - 1\}. \tag{5b}$$

Clearly $x_{\frac{N}{2}} = d$, $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$ and σ_1, σ_2 are taken to be the following

$$\sigma_1 = \min\left\{\frac{d}{2},\ 2\frac{\varepsilon}{\theta_1}\ln N\right\},\quad \sigma_2 = \min\left\{\frac{1-d}{2},\ 2\frac{\varepsilon}{\theta_2}\ln N\right\}, \tag{5c}$$

where $\theta_1 = \max\{-cu_\varepsilon(0), 1 - cu_\varepsilon(1)\}; \theta_2 = \max\{cu_\varepsilon(1), 1 + cu_\varepsilon(0)\}$. Then the fitted mesh method for problem (1) is: Find a mesh function U_ε such that

$$\varepsilon\delta^2 U_\varepsilon(x_i) + b(x_i, U_\varepsilon(x_i))DU_\varepsilon(x_i) = f(x_i) \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N, \tag{6a}$$

$$U_\varepsilon(0) = u_\varepsilon(0), \quad U_\varepsilon(1) = u_\varepsilon(1), \tag{6b}$$

$$D^- U_\varepsilon(x_{\frac{N}{2}}) = D^+ U_\varepsilon(x_{\frac{N}{2}}), \tag{6c}$$

where

$$\delta^2 Z_i = \frac{D^+ Z_i - D^- Z_i}{(x_{i+1} - x_{i-1})/2} \quad , DZ_i = \begin{cases} D^- Z_i, & i < N/2, \\ D^+ Z_i, & i > N/2, \end{cases}$$

D^+ and D^- are the standard forward and backward finite difference operators, respectively. This is a nonlinear finite difference scheme. In practice, the nonlinear system is solved using a continuation method similar to that in [3].

The same conditions required for existence of the solution of the full continuous problem are also sufficient for the existence (but not uniqueness) of the solution of the discrete nonlinear problem.

In [4], it is established that, providing N is sufficiently large and ε is sufficiently small, independently of each other, under the further implicit restriction that

$$b^2(x_i, U_\varepsilon) - 4\varepsilon c u'_\varepsilon > 0, \quad x_i \ne d, \tag{7}$$

we can prove a uniform in ε error bound at all the mesh points of the form

$$\|U_\varepsilon - u_\varepsilon\|_\Omega \le CN^{-1}(\ln N)^2, \tag{8}$$

where u_ε is the continuous solution, U_ε is a discrete solution of (6), and C is a constant independent of N and ε.

3 Robustness of the Solution Method

Example 1: For the uniform convergence result (8) to be valid, [4] requires that (4) and (7) must be satisfied. For example, if

$$c = 1, \quad \delta_1 d < -u_\varepsilon(0) < 0.1 \quad \text{and} \quad \delta_2(1-d) < u_\varepsilon(1) < 0.1$$

then the data constraints (4) and (7) in $\mathbf{C}_3$ are both satisfied. Thus a problem with

$$d = 0.25,\ \delta_2 = 0.13,\ \delta_1 < 0.4,\ 0.0975 < u_\varepsilon(1) < 0.1,\ \ -0.1 < u_\varepsilon(0) < -\delta_1/4$$

satifies these constraints. We consider a problem with u(0) = -0.09, u(1) =0.098, $\delta_2 = 0.13$ and δ_1 varying from 0.1 to 0.35. This choice for the data satisfies all three assumptions including the implicit one (7). We verify this assertion numerically by computing

$$T_\varepsilon^N(x_i) = \begin{cases} b^2(x_i, U_\varepsilon^N) - 4\varepsilon D^- U_\varepsilon^N, & x_i < d \\ b^2(x_i, U_\varepsilon^N) - 4\varepsilon D^+ U_\varepsilon^N, & x_i > d \end{cases}$$

and observing that $T_\varepsilon^N = \min_i T_\varepsilon^N(x_i) > 0$ for all values of ε and N used. The computed uniform rates of convergence p_N, using the double mesh principle and the uniform fine mesh errors E_N (see [1] for details on how these quantities are calculated) are given in Table 1, which confirm uniform convergence in this range of the data. In passing we note that as expected an upwinded scheme

N	32	64	128	256	512	1024
			$\delta_1 = 0.1$			
E_N	0.004962	0.003227	0.002017	0.001175	0.000637	0.000313
p_N	0.46	0.75	0.63	0.72	0.68	0.84
			$\delta_1 = 0.2$			
E_N	0.003583	0.002245	0.001346	0.000771	0.000413	0.000201
p_N	0.57	0.76	0.72	0.72	0.72	0.85
			$\delta_1 = 0.3$			
E_N	0.002549	0.001403	0.000809	0.000457	0.000243	0.000117
p_N	0.70	0.90	0.79	0.76	0.73	0.86
			$\delta_1 = 0.35$			
E_N	0.002205	0.001151	0.000584	0.000295	0.000155	0.000075
p_N	0.90	0.94	0.96	0.93	0.72	0.88

Table 1. Maximum errors E_N and computed rates of convergence p_N for the numerical method (5),(6) in the case of Example 1.

on a uniform mesh does not converge uniformly in ε as shown in Table 2.

Now consider the same problem with u(0) = -0.09, u(1) =0.098, $\delta_2 = 0.13$ and $\delta_1 = 0.39$. This does not satisfy (3) or (4a). However, this scheme does numerically satisfy the implicit condition (7). The results presented in Table 3 imply that the scheme is still uniformly in ε convergent.

Example 2: For the existence of a continuous solution we have the sufficient conditions (4). As an example, take

$$c = 1, \qquad u_\varepsilon(1) = 0.7,\ u_\varepsilon(0) = -0.5 \quad d = 0.25.$$

Then (3) is satisfied when $\delta_1 < 2.5 \quad \delta_2 < 1.26$. Also (4a) is satisfied when

$$\delta_1 < 2 \quad \delta_2 < \frac{2.8}{3} \approx 0.933333$$

and (4b) is satisfied when

N	32	64	128	256	512	1024
$\delta_1 = 0.1$						
E_N	0.007397	0.007215	0.007009	0.006685	0.006095	0.005002
p_N	0.02	0.01	0.00	0.00	0.00	0.00
$\delta_1 = 0.2$						
E_N	0.005258	0.004944	0.004711	0.004445	0.004038	0.003341
p_N	0.07	0.04	0.02	0.01	0.00	0.00
$\delta_1 = 0.3$						
E_N	0.003533	0.002921	0.002578	0.002332	0.002067	0.001680
p_N	0.32	0.17	0.07	0.04	0.02	0.01
$\delta_1 = 0.35$						
E_N	0.003292	0.001887	0.001488	0.001252	0.001064	0.000841
p_N	0.85	0.38	0.24	0.09	0.05	0.03

Table 2. Maximum errors E_N and computed rates of convergence p_N for scheme (6) on a uniform mesh in the case of Example 1.

$\delta_1 = 0.39$						
N	32	64	128	256	512	1024
E_N	0.002282	0.001154	0.000578	0.000283	0.000133	0.000057
p_N	0.98	0.96	0.98	0.99	0.99	1.00

Table 3. Maximum errors E_N and computed rates of convergence p_N when conditions (3) and (4a) are not satisfied.

$$\delta_1 > 1.2 \quad \text{and} \quad \delta_2 > \frac{1.36}{3} \approx 0.453333.$$

We consider various values of δ_1 which violate one or more of the conditions (3) , (4a) or (4b). Table 4 gives the conditions that are violated for a number of values of the parameter δ_1. Illustrations of the corresponding solutions are given in Figure 2, and the convergence results are given in Table 5. They show that provided the reduced solution of the problem remains monotonic increasing, the method is robust in the sense that the numerical method remains uniformly in ε convergent. When the problem ceases to be monotonic the layer type changes from a standard shock layer to an S-layer. As the S-layer grows in amplitude the nonlinear solver does not converge and thus the method ceases to be robust.

δ_1	Condition violated
0.2	(4b)
1.1	(4b)
2.0	(4a)
2.49999	(4a)
2.5	(4a), (3)
3.8	(4a) (3)

Table 4. Conditions violated by Example 2 for various values of δ_1.

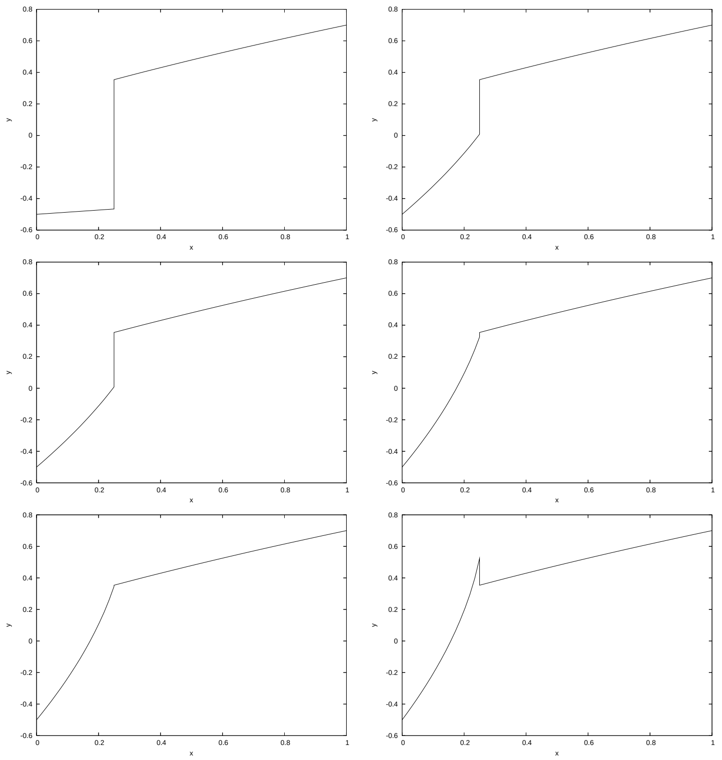

Fig. 2. Solution of (1) for problems which do not satisfy $\mathbf{C}_3$. In all these figures, $\delta_2 = 0.7,\ u(0) = -0.5,\ u(1) = 0.7,\ N = 64$ and $\varepsilon = 0.000001$. From top left to bottom right : $\delta_1 = 0.2$, 2.4999, 2.5, 3.5, 3.55, 3.9.

4 Sensitivity to the Position of the Transition points

We examine the effect of varying the fine mesh width by incorporating a constant C_* in a revised formula for σ_1 and σ_2 given by

$$\sigma_1 = \min\left\{\frac{d}{2},\ C_*\frac{\varepsilon}{\theta_1}\ln N\right\},\quad \sigma_2 = \min\left\{\frac{1-d}{2},\ C_*\frac{\varepsilon}{\theta_2}\ln N\right\}, \tag{9}$$

where $\theta_1 = \max\{-cu_\varepsilon(0), 1 - cu_\varepsilon(1)\}$, $\theta_2 = \max\{cu_\varepsilon(1), 1 + cu_\varepsilon(0)\}$ and C_* is a parameter.

Table 6 give the results for Example 2 with $\delta_1 = 1.20010$. The number of iterations are at most twice for the different examples. Thus the method is not particularly sensitive to the fine mesh width and, in fact, a choice of a value of C_* less than that of $C_* = 2$ used in [4] seems to give better performance. In the example considered here, the errors are smallest and the rate of convergence best for $C_* = 0.5$.

N	32	64	128	256	512	1024
$\delta_1 = 0.2$						
E_N	0.085977	0.070653	0.045129	0.028786	0.016281	0.008038
p_N	0.01	0.62	0.70	0.55	0.70	0.70
$\delta_1 = 0.5$						
E_N	0.081286	0.063318	0.039899	0.025084	0.014299	0.007035
p_N	0.00	0.62	0.70	0.56	0.74	0.70
$\delta_1 = 1.1$						
E_N	0.071339	0.055289	0.034691	0.021476	0.012067	0.005918
p_N	0.08	0.65	0.71	0.57	0.76	0.71
$\delta_1 = 1.2$						
E_N	0.069654	0.054195	0.033900	0.021033	0.011889	0.005824
p_N	0.08	0.65	0.71	0.57	0.75	0.71
$\delta_1 = 2.0$						
E_N	0.054596	0.044248	0.028406	0.017600	0.009784	0.004840
p_N	0.15	0.67	0.73	0.56	0.69	0.71
$\delta_1 = 2.4$						
E_N	0.045858	0.037679	0.024406	0.014925	0.008380	0.004132
p_N	0.21	0.68	0.74	0.59	0.67	0.72
$\delta_1 = 2.4999$						
E_N	0.043529	0.035851	0.023213	0.014147	0.007960	0.003927
p_N	0.23	0.67	0.74	0.60	0.68	0.72
$\delta_1 = 3.0$						
E_N	0.043328	0.025441	0.015947	0.009703	0.005490	0.002714
p_N	0.83	0.63	0.79	0.69	0.68	0.71
$\delta_1 = 3.5$						
E_N	0.075558	0.032340	0.015213	0.007286	0.003408	0.001470
p_N	1.32	1.12	1.04	1.00	0.99	0.98
$\delta_1 = 3.8$						
E_N	0.168256	0.056174	0.024782	0.011446	0.005227	0.002217
p_N	1.84	1.24	1.10	1.05	1.02	1.01

Table 5. Maximum errors E_N and computed rates of convergence p_N for the numerical method (5) in the case of Example 2.

5 Conclusions

The numerical results in this paper indicate a possible gap between the theory in [4] and what is observed in practice. As was proven in [4] the scheme (5), (6) is a parameter-uniform scheme under the conditions (4) and (7). However these sufficient conditions appear to be overly restrictive, since, in practice, the numerical approximations appear to converge for a wider range of data. In any attempt to extend the theory in [4] to a wider class of problems, a reasonable constraint on the data to aim for (in place of (4)) would be that the reduced solution is monotonic increasing, which is a necessary condition to exclude S-layers from appearing in the solution of (1).

The implicit condition (7) is not satisfied for some of the examples presented here, while the numerical approximations still converge uniformly in ε. When the constraint (7) is violated it appears that $T_\varepsilon^N(x_i) < 0$ in a particular neighborhood of the point d and not at the transition points between the fine and coarse mesh. Proving convergence without (7) being satisified would require a method of proof other than the maximum principle arguments used

N	32	64	128	256	512	1024
$C_* = 0.125$						
E_N	0.077109	0.063909	0.052342	0.040499	0.028576	0.017859
p_N	0.37	0.34	0.27	0.24	0.26	0.27
$C_* = 0.25$						
E_N	0.055713	0.034658	0.020660	0.011906	0.006556	0.003274
p_N	0.70	0.68	0.71	0.71	0.71	0.70
$C_* = 0.5$						
E_N	0.039241	0.021406	0.012181	0.006681	0.003483	0.001645
p_N	0.81	0.89	0.79	0.80	0.82	0.78
$C_* = 1.0$						
E_N	0.052324	0.033291	0.020706	0.011990	0.006454	0.003099
p_N	0.23	0.79	0.68	0.73	0.77	0.76
$C_* = 2.0$						
N	32	64	128	256	512	1024
E_N	0.069652	0.054194	0.033899	0.021033	0.011889	0.005824
p_N	0.08	0.65	0.71	0.57	0.75	0.71

Table 6. Maximum errors E_N and computed rates of convergence p_N for various choices of the transition point in the case of Example 2 with $\delta_1 = 1.20010$.

in [4]. These numerical results also suggest that a different finite difference equation (other than continuity of the discrete first derivative) at the point of the discontinuity d may ensure that $T_\varepsilon^N > 0$, which in turn might improve the performance of the scheme and also assist in extending the scope of the current theory.

References

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